THE FOURTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The fourteenth regular meeting of the Southwestern Section of the American Mathematical Society was held at Washington University on Saturday, November 26, 1921. The attendance included the following twenty members:

Ammerman, Ashton, Dunkel, W. W. Hart, Hedrick, Ingold, Lefschetz, Luby, Lytle, McKelvey, U. G. Mitchell, Rider, Roever, Slaught, Edwin R. Smith, Eugene Stephens, Stouffer, E. A. Weeks, Westfall, J. M. Young.

The afternoon session was a joint meeting with the Missouri Section of the Mathematical Association of America. Professor Hedrick occupied the chair during the morning session and was relieved during the afternoon session by Professor Ingold, chairman of the Missouri Section.

It was voted to hold the next meeting of the Southwestern Section at the University of Kansas. The following program committee was elected: Professors Ashton (chairman), Brenke, and Stouffer (secretary).

The titles and abstracts of the papers read at this meeting follow below. The papers of Dr. Kendall and Professor Chittenden were read by title.

1. Dr. Claribel Kendall: *Certain congruences determined by a given surface.*

In this paper a surface is considered as defined by a pair of partial differential equations of the second order as is done by Professor Wilczynski in his first memoir on the projective differential geometry of curved surfaces (TRANSACTIONS OF THIS SOCIETY, vol. 8). A line l is determined by two points which are defined with respect to a local tetrahedron of reference and in terms of the parameters of the surface. Thus a line is given for every point on the surface and hence a congruence is determined. Relative to this general congruence, formulas are found (1) for the torsal curves on the surface, i.e., the curves which give the developables of the congruence, (2)

for a new class of curves which are called the guide curves of the congruence, (3) for the focal points of the line l. These formulas are applied to special congruences and various relations among the lines generating these congruences are found.

2. Professor Otto Dunkel: *The direct determination of the minimum area between a curve and its caustic.*

In the WASHINGTON UNIVERSITY STUDIES, SCIENTIFIC SERIES, vol. 8, no. 2, pp. 183–194, necessary and sufficient conditions were derived for the existence of a minimum area by methods similar to the usual methods of the calculus of variations. The present treatment by Professor Dunkel is similar to the one given by him for the solution of Euler's problem (AMERICAN MATHEMATICAL MONTHLY, vol. 28, pp. 15–19) in that advantage is taken of the fact that the integrand may be reduced to a square and that the problem is solved by elementary means without the use of the general theory of the calculus of variations. The solution of this problem differs from that of Euler's problem, notably in that most of the results are obtained by reductions to complete squares.

3. Professors E. R. Hedrick, Louis Ingold, and W. D. A. Westfall: *Classification of non-analytic functions.*

In this paper Beltrami's extensions of the Cauchy-Riemann equations are made the basis for a classification of all non-analytic functions. Two functions belong to the same class if the fundamental quantities E, F, G, for the two functions are proportional. Some properties of functions belonging to the same class are then developed.

4. Professor S. Lefschetz: *A topological group of algebraic varieties.*

The group in question is the transformation group G of the $(d-1)$-dimensional cycles of a linear pencil of hypersurfaces of a d-dimensional variety. The singularities being ordinary, the operations of G acting upon a given cycle are completely defined by means of Kronecker characteristics (the algebraic number of intersections of cycles). For d even the group is of the same type as the Picard group, defined by him for $d = 2$. For d odd the fundamental substitutions are of order *two*, G being nevertheless infinite.

5. Professor S. Lefschetz: *On multiply periodic functions.*

Given a period matrix for $2p$-ply periodic meromorphic functions, we may define for the variables a fundamental domain in the form of a $2p$-dimensional generalized ring. The zeros common to $p - 1$ suitable periodic functions define a two-cycle of the ring. This cycle can be readily expressed in terms of those of a fundamental set. Several results are obtained, the most important being the Picard-Poincaré existence theorem for the matrix, and the classification of all intermediary functions in terms of a certain fundamental set.

6. Professor W. H. Roever: *An application of the theorem of Jacobi.*

In this paper Professor Roever applies the theorem of Jacobi* to the integration of the differential equations of motion of a particle referred to a rotating frame of reference, when a constant force acts parallel to the axis of rotation.

7. Professor E. B. Stouffer: *On curves on singular ruled surfaces.*

A ruled surface in a space of five dimensions is said to be singular if every set of three consecutive generators lies in a space of four dimensions. On every such singular ruled surface there is a curve along which tangents to the surface intersect three consecutive generators. Moreover, this curve is unique unless the ruled surface lies entirely in a space of three dimensions. A system of three linear homogeneous differential equations is used for the study of these forms.

8. Professor E. W. Chittenden: *On the division of a plane by a set of points.*

The following theorem is proved. Suppose K is a plane point set, S is the set of all points of the plane, while $S - K = S_1 + S_2$, where S_1 and S_2 are two mutually exclusive domains such that every point of K is a common limit point of S_1 and S_2. Then the set K is closed and connected. That K is connected was shown by J. R. Kline under the added restriction that K be connected "im kleinen" (TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 452, Lemma B).

E. B. STOUFFER,
Secretary of the Section.

*For a statement of this theorem see, for instance, Appell, *Traité de Mécanique Rationnelle*, 4th edition, vol. 1, p. 560.

THE TWENTY-EIGHTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The twenty-eighth annual meeting of the Society and the forty-eighth regular meeting of the Chicago Section were held at the University of Toronto on Wednesday and Thursday, December 28–29, 1921, in affiliation with the Convocation week meetings of the American Association for the Advancement of Science. Dormitory and eating accommodations for the mathematical group were furnished by the University of Toronto, Knox College being almost wholly given up to this purpose. Hart House was the center of a generous hospitality which filled the hours outside the scientific program with a delightful and varied entertainment.

On Thursday morning there was a joint session with Sections B and C of the American Association for the Advancement of Science and with the American Physical Society, and on Thursday afternoon a joint session with Section A and the Mathematical Association of America. At a special Convocation, the University of Toronto conferred on Professor E. H. Moore, as president of the American Association for the Advancement of Science, the degree of Doctor of Science, *honoris causa*.

The attendance included the following eighty-three members of the Society: Archibald, G. N. Armstrong, Atchison, Bacon, Beatty, Beetle, Bliss, Bradshaw, R. W. Burgess, Cairns, H. K. Cummings, L. D. Cummings, Currier, Curtiss, Dadourian, DeLury, Denton, L. L. Dines, Dostal, Dresden, Peter Field, Fields, Findlay, Finkel, Focke, W. B. Ford, J. L. Gibson, Gillespie, Glashan, Glenn, Glover, Gummer, Haskins, Hazlett, E. R. Hedrick, Howe, Huntington, Hurwitz, Ingraham, Karpinski, Kingston, Lennes, F. P. Lewis, Marshall, T. E. Mason, G. A. Miller, Norman Miller, Mirick, E. H. Moore, F. R. Moulton, Newkirk, Olson, F. W. Owens, H. B. Owens, Leigh Page, Pell, Phillips, Pitcher, Pounder, A. V.

Richardson, R. G. D. Richardson, H. L. Rietz, Robison, E. D. Roe, Roever, Sheppard, Sinclair, Slaught, C. E. Smith, Edwin R. Smith, P. F. Smith, Synge, Uhler, Veblen, Waddell, Watkeys, Webster, Wiener, Wilder, W. L. G. Williams, A. H. Wilson, E. B. Wilson, J. M. Young.

President Bliss occupied the chair at the regular meetings on Wednesday, being relieved at the morning session by Professor P. F. Smith and at the afternoon session by Professor C. N. Haskins. The joint session of Thursday morning was presided over by Professors G. W. Stewart and W. D. Harkins, Vice-Presidents of Sections B and C respectively, and that of Thursday afternoon by Professor Oswald Veblen, Vice-President of Section A.

The Council announced the election of the following sixty-two persons to membership in the Society:

Professor William Cyrus Bartol, Bucknell University;
Professor William John Bauduit, Howard University;
Mr. Benjamin Berman, University of Maryland;
Miss Florence Lucile Black, University of Kansas;
Professor Edmond Wesley Bowler, New Hampshire College;
Professor John Wentworth Clawson, Ursinus College;
Professor Robert Cameron Colwell, Geneva College;
Professor Charles Winthrop Crockett, Rensselaer Polytechnic Institute;
Mr. Herbert Kimball Cummings, Brown University;
Professor Lloyd Slote Dancey, Carroll College;
Mr. William Whitfield Elliott, Cornell University;
Professor John Thomas Erwin, George Washington University;
Professor Earl Frederick Farnau, University of Cincinnati;
Professor Forest Almos Foraker, University of Pittsburgh;
Mr. William McKinley Gafafer, Columbia University;
Mr. Harry Merrill Gehman, University of Pennsylvania;
Professor Albert Henry Steward Gillson, McGill University;
Professor J. W. Harrell, Baylor University;
Professor George Abram Harter, University of Delaware;
Miss Eugenie Caroline Hausle, Columbia University;
Professor Ruby Usher Hightower, Shorter College;
Mr. Van Buren Hinsch, Missouri School of Mines;
Mr. Frank Adin Joy, New Hampshire College;
Professor Oscar Godfrey Lawless, Talladega College;
Mr. Maurice Moses Levita, Temple University;
Mr. Thomas Richard Long, University of Rochester;
Professor Frederick Milton McGaw, Cornell College;
Professor James McGiffert, Rensselaer Polytechnic Institute;

Mrs. Martha (Macdonald) McKelvey, Iowa State College;
Professor Neil Bruce MacLean, University of Manitoba;
Miss Ida Caroline Millay, New Hampshire College;
Professor Clifford Newton Mills, Heidelberg University;
Professor Eugenie Maria Morenus, Sweet Briar College;
Dr. Hermance Mullemeister, University of Washington;
Dean Albert O'Brien, O. F. M., St. Bonaventure's College;
Mr. Frederick Allen Parker, West Virginia Collegiate Institute;
Professor Annie McKinnie Pegram, Greensboro College;
Mr. Cecil Glenn Phipps, University of Minnesota;
Professor Joseph Romeo Plante, Viator College;
Mr. Louis Gordon Pooler, Columbia University;
Mr. Abraham Press, Patent Section, Air Service, War Department;
Professor George Edwin Ramsdell, Bates College;
Dean Otis Everett Randall, Brown University;
Professor Joseph Marshall Rankin, College of Idaho;
Mr. George Emil Raynor, Princeton University;
Professor Arthur Vernon Richardson, Bishop's College;
Professor Tom Gladstone Rodgers, New Mexico Normal University;
Professor Sidney Archie Rowland, Jr., Union College;
Professor Henry Alford Ruger, Columbia University;
Professor Joseph Seidlin, Alfred University;
Mr. Leonard Shaffer, New Hampshire College;
Professor Rowland Alfred Sheets, Denison University;
Professor Wilfred H. Shirk, University of Buffalo;
Mr. George Yale Sosnow, Prudential Insurance Company;
Professor Guy Greene Speeker, Michigan Agricultural College;
Professor Mary Cass Spencer, Newcomb College, Tulane University;
Mr. Joe Ichino Tanoue, Columbia University;
Mr. Louis Weisner, Columbia University;
Professor John Jefferson Wheeler, University of Kansas;
Mr. Daniel Everett Whitford, University of Rochester;
Professor Hugh Carey Willett, University of Southern California;
Miss Ruby Willis, Wellesley College.

The invitation extended by the University of Rochester to hold the summer meeting of 1922 at that institution was accepted, and Professors Gale (chairman), Hildebrandt, Metzler, Watkeys, and the Secretary were constituted a committee on arrangements. Professor O. D. Kellogg and the Secretary were appointed as representatives of the Society on the Council of the American Association for the Advancement of Science for 1922.

The Council voted to request that the space occupied in the BULLETIN by an abstract of a paper presented to the Society

should be limited to the equivalent of two hundred words of text, and that display typography be avoided.

The total membership of the Society is now 1005, including 85 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 420; the number of papers read was 175. The number of members attending at least one meeting was 273. At the annual election 169 votes were cast.

The reports of the Treasurer and of the Auditors (Mr. S. A. Joffe and Professor P. H. Linehan) were received, showing a balance of $10,604.22, including the Life Membership Fund, which now amounts to $7,528.87. Sales of the Society's publications during the year amounted to $3,222.16.

The Library now contains 6,014 volumes, excluding some 500 unbound dissertations, now being catalogued.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor R. D. CARMICHAEL,
	Professor D. E. SMITH.
Secretary,	Professor R. G. D. RICHARDSON.
Treasurer,	Professor W. B. FITE.
Librarian,	Professor R. C. ARCHIBALD.

Committee of Publication

Professors E. R. HEDRICK, W. A. HURWITZ, J. W. YOUNG.

Members of the Council to serve until December, 1924

Professor J. W. ALEXANDER,	Professor L. L. DINES,
Professor HENRY BLUMBERG,	Professor F. R. SHARPE.

At the end of the afternoon session of Wednesday, there was a business meeting of the Chicago Section. The officers elected for the next biennium are Professor A. B. Coble, Chairman; Professor Arnold Dresden, Secretary; and Professor E. W. Chittenden.

The committee on the symposium for April, 1922, reported that the symposium lecture would be given by Professor A. B. Coble on *Cremona transformations and applications to algebra, geometry, and modular functions.*

The meeting of the Society immediately preceded that of the Mathematical Association of America. A very pleasant occasion was the joint dinner, on Friday evening, of the various mathematical, physical, and astronomical associations. At the afternoon session of Wednesday it was voted to express the thanks of the Society for the hospitality of the Royal Canadian Institution and the University of Toronto, and for the entertainment furnished by the local committee of the American Association for the Advancement of Science.

On Thursday morning the program was as follows:

I. *Atomic nuclei and extra-nuclear electronic configuration,* by Professor J. C. McLennan, retiring Vice-President of Section B.

II. Symposium on quantum theory: for Section C, Dr. R. C. Tolman; for the American Mathematical Society, Professor H. B. Phillips; for the American Physical Society, Dr. Saul Dushman.

At the joint session on Thursday afternoon, the following addresses were given:

I. *A mechanical analogy in the theory of equations,* by Professor D. R. Curtiss, retiring Vice-President of Section A.

II. *The research information service of the National Research Council,* by Professor R. M. Yerkes, of the National Research Council.

III. *Subsidy funds for mathematical projects,* by Professor H. E. Slaught.

IV. *Algebraic guides to transcendental problems,* by Professor R. D. Carmichael, retiring Chairman of the Chicago Section.

In the absence of Professor Carmichael, an abstract of his paper was given by Professor Dresden; the paper appears in full in this number of the BULLETIN. Professor Slaught's paper has appeared in SCIENCE.

Titles and abstracts of the papers read at the regular sessions on Wednesday follow below. Professor Schwatt's first paper was read by Professor Pounder. The papers of Dr. Zeldin, Professors Crum, Forsyth, and Emch, Dr. Hille, Professors Lipka, MacNeish, Everett, Dodd, Wilczynski, and Glenn, Professor Schwatt's second paper and that of Dr. Wiener and Dr. Walsh were read by title.

1. Professor C. E. Wilder: *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions.*

The differential geometry of $(n-1)$-dimensional varieties in an n-dimensional space has long been developed. More recently several authors have treated the two-dimensional manifold in n dimensions.

In this paper Professor Wilder takes up the theory of the intermediate cases and generalizes certain theorems and methods of ordinary surface theory. The method used is that of the absolute calculus of Ricci combined with n-dimensional vector analysis.

2. Professor C. E. Wilder: *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions. Second paper.*

The m-dimensional manifold in n dimensions not only has extension in different directions as has the surface in ordinary space, but also it winds its tortuous way through n dimensions as does the twisted curve in three dimensions. In this paper certain facts of curve theory are generalized to an m-dimensional manifold in n dimensions.

3. Mr. M. H. Ingraham: *A modification of Peano's postulates for positive integers.*

In this paper is studied the complete independence of Peano's postulates for positive integers, and of Padoa's modification of these postulates. This study is made by means of their complete existential theory, as introduced by E. H. Moore in his New Haven Colloquium Lectures, 1906. It is proved that these postulates are not completely independent. A modification of these systems of postulates is given which is completely independent.

4. Professors L. P. Eisenhart and Oswald Veblen: *The Riemann geometry and its generalizations.*

The authors determine the conditions under which the n-dimensional geometry of a system of *paths* defined by the solution of a system of differential equations

$$\frac{d^2x^i}{ds^2} = \Gamma_{jk}{}^i \frac{dx^j}{ds}\frac{dx^k}{ds},$$

in which the Γ's are functions of $x^1, x^2, \cdots, x^n$ subject only to the condition $\Gamma_{jk}{}^i = \Gamma_{kj}{}^i$ reduces to the Riemann geometry determined by a quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j.$$

The conditions obtained are such as to indicate that the metric introduced by Weyl as the basis of combined electromagnetic and gravitation theory is next to the Riemann metric in order of simplicity. The paper appears in the PROCEEDINGS OF THE NATIONAL ACADEMY, February, 1922.

5. Professor R. W. Burgess: *The problem of apportionment. The method of the weighted geometric mean.*

If N representatives are to be apportioned among several states of total population P, the true quota for a state of population A is $\alpha = AN/P$, ordinarily a fraction. The number of representatives actually assigned the state must be an integer, say a. The ratio $r = a/\alpha$ or α/a, whichever is greater than unity, is fundamental in the consideration of the problem. Professor Burgess suggests a method of apportionment based on the following single postulate: That apportionment is most satisfactory for which the continued product of terms of the type r^A, one term for each state, is smallest. This in effect takes the ratio for each inhabitant of each state, and uses the product of all such ratios as the measure of unfairness. If the exponents of the r were in each case unity, i.e., if the ratio were taken once for each state rather than once for each individual, this method would, in the usual case, become equivalent to those proposed by Dr. Hill and Professor Huntington. As in this new method the quantity $[\alpha/\sqrt{a(a+1)}]^A$ plays about the same rôle as does $A/\sqrt{a(a+1)}$ in Professor Huntington's method, and $A/(a+1/2)$ in Professor Willcox's method, it may be called the method of the weighted geometric mean.

6. Professor E. V. Huntington: *Necessary and sufficient conditions in the problem of apportionment.*

The present paper compares the method of major fractions and the method of equal proportions by analyzing all the known conditions or tests (including several new ones) which have been suggested in favor of either of them.*

The real question is believed to be one of simple equality between each state and every other state, here and now; and the tests based on this idea point clearly to the method of equal proportions.

Tests which attempt to compare each state directly with a norm for the whole country are found to be, in general, indeterminate or unworkable. Tests based on the idea of a total or average deviation from the norm lead to so many different methods that a choice among them appears difficult to make, especially when the averaging process is extended over unlimited time.

7. Dr. S. D. Zeldin: *Commutativity of contact transformations of mechanics.*

It is well known that with every conservative dynamical system, defined by its potential energy (a function of the generalized coordinates) and its kinetic energy (a quadratic function of the generalized velocity components), an infinitesimal contact transformation is associated whose characteristic function is of a special type. In this paper Dr. Zeldin proves that if two contact transformations associated with dynamical systems, defined as above, are independent, they cannot be commutative.

8. Professor G. A. Miller: *Substitutions commutative with every substitution of an intransitive group.*

This paper appears in full in this number of the BULLETIN.

* For full bibliography up to the middle of 1921 (with special reference to the work of J. A. Hill) see *A new method of apportionment*, by E. V. Huntington, QUARTERLY PUBLICATION OF THE AMERICAN STATISTICAL ASSOCIATION, September, 1921. See also *Report of the Advisory Committee to the Director of the Census*, ibid., December, 1921, and a forthcoming paper by F. W. Owens in the same journal. Letters from E. V. Huntington and W. F. Willcox have appeared in the NEW YORK TIMES for Oct. 18, Oct. 23, Nov. 13, and Dec. 11, 1921. Recent papers have been presented to the American Mathematical Society by E. V. Huntington, Sept. 8, and R. W. Burgess, Dec. 28, 1921.

9. Professor G. A. Miller: *Seeming contradictions in the theory of groups.*

There are at least three distinct definitions of the term "same group" in common use in the theory of groups of finite order: (1) two substitution groups are the same if and only if one can be transformed into the other by a substitution; (2) two substitution groups or two abstract groups are the same if and only if a simple isomorphism can be established between their substitutions or operators; (3) two groups are the same if and only if one is obtained from the other by an interchange of operators. Various seeming contradictions that arise from the use of these different definitions are pointed out.

10. Professor W. A. Hurwitz: *Convergence-factors in Cesàro-summable series.*

Sufficient conditions for the theorem on convergence-factors for Cesàro-summable series have been given by C. N. Moore, Hardy, Bromwich and Chapman. Results essentially identical with part of the theorem have been proved by Schur, Carmichael and others. By means of a slight further generalization of the Silverman-Toeplitz criterion as extended by Hildebrandt and Carmichael, the author establishes necessary and sufficient conditions for the convergence-factor theorem in the case of Cesàro-summability of any order, real or complex, with positive real component.

11. Dr. W. L. Crum: *Note on the determination of the rectilinear secular trend of an ordered series of statistical relatives.*

In the analysis of certain historical series, the question arises whether, in determining the position of the line by the method of least squares, the relative deviations should not be used rather than the absolute deviations. The paper determines the line, to a first approximation, by using relative deviations, and shows that the divergence from the line as ordinarily determined is not likely to be of importance for practical applications; but there are significant theoretical consequences.

12. Professor C. H. Forsyth: *Provisions for depreciation based directly upon appraisal.*

Practically all methods of computing the depreciation charge succeed in providing a fund which will accumulate and ultimately meet the expense of replacing the property at the end

of its lifetime, but no definite relation is maintained between the amount in the fund and the total amount of depreciation at any intermediate date. A method of computing the annual depreciation charge is given wherein estimates of periodic depreciation by an efficient appraiser are the foundation and accumulations of interest upon the fund are employed to reduce later charges. The scheme then insures a constant equality between the true wearing value and the theoretical wearing value with a degree of accuracy which will depend upon the reliability of the appraiser's estimates.

13. Professor Arnold Emch: *Plane algebraic curves invariant under a given quadratic Cremona transformation.*

The transformation which the author chooses in this paper has the well known simple form $\rho x_1' = x_2x_3$, $\rho x_2' = x_3x_1$, $\rho x_3' = x_1x_2$, with $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, A_3 $(0, 0, 1)$ as the base, and $B(1, 1, 1)$, $B_1(-1, 1, 1)$, $B_2(1, -1, 1)$, $B_3(1, 1, -1)$ as the invariant points. The following theorem is proved by means of a net of invariant cubics: Let C_n be an invariant n-ic under the given transformation with ordinary multiple points of possible orders $\mu_1, \mu_2, \mu_3, \nu, \nu_1, \nu_2, \nu_3$ at the A's and B's respectively. Then $\mu_1 + \mu_2 + \mu_3 = n$, and the lines joining corresponding points of the transformation on the C_n envelope a curve $\mathfrak{C}_m$ of class $m = n - \frac{1}{2}(\nu + \nu_1 + \nu_2 + \nu_3)$. Conversely, the locus of pairs of corresponding points on the tangents of any curve $\mathfrak{C}_m$ is a certain invariant C_n. All invariant C_n's may be generated in this manner. When the $\mathfrak{C}_m$ is in a general position, then C_n is of order $n = 3m$. In case of a rational $\mathfrak{C}_m$, the C_n is, in general, hyperelliptic.

As an application Professor Emch studies hyperelliptic sextics, and rational invariant curves.

14. Professor Joseph Lipka: *Canonical systems and the general problem of dynamics.*

In this paper, Professor Lipka first derives the most general infinitesimal homogeneous contact transformations in $2n$ variables, $x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n$, connected by a relation $H(x_1, \cdots, x_n, p_1, \cdots, p_n) =$ constant, which leave the Pfaffian $\Sigma_i p_i dx_i$ invariant. Interpreting this result properly in a metric space defined by $ds^2 = \Sigma_{ik} a_{ik} dx_i dx_k$, the equations of the transformation reduce to the canonical equations of the dynamical trajectories in a conservative field of force in a curved space of n dimensions. We thus get a very elegant

and very brief proof of the characteristic property of orthogonality of a system of dynamical trajectories (or of any natural family of curves)—including both the direct theorem of Lipschitz* and its converse as given by the writer.†

15. Professor H. F. MacNeish: *Euler squares.*

Euler proposed the following problem called "the problem of the 36 officers": Six officers of different rank are chosen from each of six different regiments. It is required to arrange them in a solid square so that no two officers of the same rank or of the same regiment shall be in the same row or the same column. This is equivalent to the problem of arranging in a square array the 36 distinct two-digit numbers with digits 1, 2, 3, $\cdots$, 6 so that there are no repetitions in the columns or in the rows either in the units' digits or in the tens' digits. This array is called an Euler square of order 6, degree 2 and index 6, 2. Euler stated without proof that the solution is impossible for order $n \equiv 2 \pmod 4$. By associating the Euler square with a geometric configuration, the author gives a proof by analysis situs. Methods are given for constructing Euler squares by means of substitution groups and by combining those of lower order. Euler squares are applied to the problem of making schedules for tournaments.

16. Professor H. S. Everett: *The expression of general forms as determinants whose elements are forms.* Preliminary report.

The author obtains the following theorems: (1) If a general form of degree ρr in $n \geqq 2$ variables has more than $[n(n+1)\cdots(n+\rho-1)/\rho! - 2]r^2 + 2$ terms, it cannot be expressed as a determinant of order $r \geqq 2$ whose elements are forms of degree $\rho \geqq 2$ in the n variables. (2) A general form of degree ρr in $n \geqq 2$ variables cannot be so expressed if $n > 3$.

These theorems are extensions of similar theorems in a paper by L. E. Dickson (TRANSACTIONS OF THIS SOCIETY, April, 1921), in which general forms expressible as a determinant with *linear* elements are completely determined.

17. Professor E. L. Dodd: *The arithmetic mean of the least and greatest of n measurements.*

Let $P(x)$ be the probability that the error of a measurement will be algebraically less than x, with $P(b) = 1$, b finite or

* JOURNAL FÜR MATHEMATIK, vol. 74 (1869).

† PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 55, No. 7, June, 1920.

$+\infty$; and let $P(x)$ have a continuous derivative $\phi(x)$. Let $\psi(z)$ be the probability density of the error z of the arithmetic mean of the least and greatest of n measurements; i.e., $\psi(z)dz$ is the probability that this error lies between z and $z + dz$. Then

$$\psi(z) = 2n(n-1)\int_a^b [P(x) - P(2z-x)]^{n-2}\phi(x)\phi(2z-x)dx.$$

Suppose that the error curve $y = \phi(x)$ for the individual measurements meets the X-axis perpendicularly at $\pm b$, like certain Pearson curves, and that the slope of this curve is numerically greater than any given G in some vicinity of $\pm b$. The curve $y = \phi(x)$, however, need not be symmetrical. Then, asymptotically for large n, the probability that the arithmetic mean of the least and greatest measurement will differ from the true value by less than e is greater than the probability that the arithmetic mean of all the measurements will so differ. Again, if $\phi(x) = c$, a constant in $(-1/2c, 1/2c)$, the arithmetic mean of the least and greatest measurement is better than the arithmetic mean of all the measurements.

18. Dr. Einar Hille: *Convex distribution of the zeros of Sturm-Liouville functions.*

Dr. Hille considers the zeros of a solution of a differential equation $w'' + G(z)w = 0$, in a convex region B in which $G(z)$ is analytic and $\vartheta_2 \leqq \arg G(z) \leqq \vartheta_1$, $(\vartheta_1 - \vartheta_2 < 2\pi)$. It is shown that all the zeros of $w(z)w'(z)$ in B, where $w(z)$ is a solution, can be joined by a polygonal line p_0, composed of oriented line-segments, each of which forms an angle θ_P with the positive real axis subject to the condition $-\vartheta_1/2 < \theta_P < -\vartheta_2/2$. Furthermore, let the zeros a_1, a_2, $\cdots$, a_n be numbered in the order in which they occur on p_0. If $\arg G(z)$ is never increasing along a certain set of polygonal lines of the same type as p_0, then $\arg (a_{\nu+1} - a_\nu)$ is a never-decreasing set of angles. In this case the points (a_ν) form the vertices of a convex polygon. Finally a *zero-free* region is determined for a given particular solution; the form of this region depends only on the argument of $G(z)$.

19. Professor D. R. Curtiss: *On Kellogg's diophantine problem.*

Professor Kellogg has proposed the problem of determining the maximum x that can occur in a solution in positive integers of the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1.$$

In a recent paper (AMERICAN MATHEMATICAL MONTHLY, vol. 28, p. 300) he gives reasons for believing that the maximum x is u_n, where $u_1=1$, and $u_{k+1}=u_k(u_k+1)$. Professor Curtiss proves this by solving the above equation for x_n and showing that the maximum value of the resulting expression for integral values of $x_1, x_2, \cdots, x_{n-1}$ which make it positive and finite but not necessarily integral, is u_n as defined above.

20. Dr. J. S. C. Glashan: *The isodyadic quintic equation.*

The object of this paper is to enable one to write down any number of isodyadic quintics independently of any knowledge of their roots. This is done by equating to zero the constant term of the diacrinic sextic given in the author's paper, *On the determination and solution of the metacyclic quintic equations with rational coefficients*, AMERICAN JOURNAL, vol. 23, p. 49.

21. Dr. J. S. C. Glashan: *On the isodyadic septimic equations.*

The object of this paper is to determine whether any given septimic equation is isodyadic, and, if it be so, to solve it. The general discussion is followed by application to the investigation and solution of four equations with numerically assigned coefficients. Following these examples there are given the solutions of the adyadic septimics and De Moivre monodyadic septimic.

22. Professor C. F. Gummer: *Criteria for relative root distributions.*

In a paper presented in September, 1920, the author developed a rational method to determine the mutual arrangement of the real roots of a system of real polynomials. In the case where $g(x)$ has roots $\beta_1, \beta_2, \cdots, \beta_q$, all real and distinct, and arranged in descending order, it was found, under reasonable restrictions, that the number of roots of $f(x)$ between β_r and β_{r+1} could be evaluated as the coefficient of t^r in a generating function of the form $\sum_{s=1}^{q} P_s(1+t)^{q-s}(1-t)^s$. The constants P_s were there determined by means of the Sturmian functions of $g(x)$. In the present paper it is shown that P_s is the signature of a certain quadratic form whose coefficients are symmetric functions of the roots of each of the polynomials f and g. No use is made of Sturmian functions. When g has k pairs of imaginary roots, the same rule will determine the

proper generating function after removal of a certain extraneous factor. A modification of this method is also given which, while depending on a less simple quadratic form, is more easily connected with the coefficients of f and g.

23. Professor Samuel Beatty: *The algebraic theory of algebraic functions.*

The general aim kept in view in the paper is to attain the simplicity and flexibility of treatment implied in deriving properties relative to a given basis from those relative to certain appropriate bases, the study of which presents less difficulty. Use is made of order-numbers of a certain type of adjointness relative to a given value of the variable. A lower bound is obtained for the number of linearly independent reduced forms of rational functions built on certain bases relative to a given value of the variable and containing none but negative powers of the element. Upper and lower bounds are obtained for the number of linearly independent reduced forms of rational functions built on a basis,—in the latter case, a basis of a certain type. The proof of the complementary theorem is effected by noting that were it to fail in any given case certain of the numbers obtained as lower bounds would not be such. The complementary theorem is used to obtain the number of conditions applicable to the reduced form of a rational function of a certain general type to build it on a given basis relative to a given value of the variable.

24. Professor I. R. Pounder: *An algebraic proof of the existence of the branches of an algebraic function.*

Under the assumption that the repeated application of Newton's process fails to separate the branches of an algebraic function y, subject to the relation $f(x, y) = 0$, this paper shows that there exists an expression $P(x^{1/\lambda}) + x^{t/\lambda}y'$, where P indicates a polynomial, which when substituted for y in $f(x, y)$ and $f_y(x, y)$ makes both the resulting expressions divisible by an arbitrarily high power of x. This being impossible when $f(x, y)$ contains no repeated factor of the same type as itself, the validity of Newton's process is established.

25. Professor E. H. Moore: *On the determinant of an hermitian matrix of quaternionic elements.*

Consider a square array or matrix $K_n \equiv ||a_{fg}||$ of n^2

quaternions a_{fg}, $(f, g = 1, \cdots, n)$. The matrix K_n is hermitian in case $\bar{a}_{fg} = a_{gf}$. Here for a quaternion

$$q \equiv c_0 + c_1 i + c_2 j + c_3 k, \quad (c_0, \cdots, c_3 \text{ real numbers})$$

$\bar{q}$ denotes the conjugate quaternion. In view of the non-commutativity of multiplication of quaternions there is probably no very useful definition of the determinant of an unconditioned quaternionic matrix K_n. However, for an hermitian quaternionic matrix K_n a useful determinant, det. K_n, is definable as follows, by suitable determination of the order of the n factors of the $n!$ terms t of the formal classical det. K_n. Let the factors of a term t be arranged in closed cycles $(f_1 f_2 \cdots f_k) \equiv a_{f_1 f_2} a_{f_2 f_3} \cdots a_{f_k f_1}$ of order k $(1 \leqq k \leqq n)$ with distinct indices $f_1, \cdots, f_k$ of $1, \cdots, n$, of which the leading index f_1 is the smallest. Let the cycles of the term t be arranged according to increasing orders k, those of the same order k being arranged according to increasing initial indices f_1. Then the quaternionic det. K_n is the formal det. K_n with this arrangement of the factors of each term. As an application of this definition, consider n m-partite quaternions or m-ary quaternionic vectors $\xi_g \equiv (b_{1g}, \cdots, b_{mg})$ $(g = 1, \cdots, n)$ and set $a_{fg} \equiv \bar{b}_{1f} b_{1g} + \cdots + \bar{b}_{mf} b_{mg}$ $(f, g = 1, \cdots, n)$. Then the matrix $K_n \equiv ||a_{fg}||$ is hermitian, and the vanishing of det. K_n is a necessary and sufficient condition that the vectors $\xi_1, \cdots, \xi_n$ be linearly dependent on the right, that is, that $a_1, \cdots, a_n$ not all 0 exist such that $\xi_1 a_1 + \cdots + \xi_n a_n = 0$.

26. Professor E. J. Wilczynski: *Some properties of the surfaces which represent the real and imaginary components of a function of a complex variable.*

The surfaces obtained by plotting separately the real and imaginary components, $u(x, y)$ and $v(x, y)$, of a function $w = u + iv$ of a complex variable $z = x + iy$, possess notable geometric properties which seem to have escaped notice so far. Professor Wilczynski calls attention to some of these: they are connected with the (imaginary) conjugate nets corresponding, on the u-surface and the v-surface, to the null-lines of the xy-plane, the (real) associate conjugate nets, and the (real) asymptotic nets of the two surfaces, where the term associate conjugate net is used in the sense of Green; i.e., at any surface point the tangents of two associate conjugate nets divide each other harmonically. The asymptotic net of the u-surface projects into an orthogonal isothermal net in the xy-plane

which is at the same time the projection of the above-mentioned real conjugate net of the v-surface, and similarly for the asymptotic net of the v-surface. The two orthogonal nets obtained in this way in the xy-plane are so related that each bisects the angles determined by the other.

27. Professor O. E. Glenn: *Note on differential invariants.*

This note relates to differential parameters for a quaternary differential form, with the condition

$$dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2,$$

and the case where the characteristic equation determining a set of covariant differentials is a reciprocal equation.

28. Professor Louise D. Cummings: *Hesse's associated points and the Weddle surface.*

The theory of eight associated points from the geometric standpoint is already classic. The purpose of the present paper is to define a point by a symmetric covariant of seven other points, to establish from this equation its geometric properties, and eventually to show the point to be identical with Hesse's eighth point.

29. Professor I. J. Schwatt: *Some of the principles of the operation with series applied to a partial fraction problem.*

By means of some of the principles of the operation with series Professor Schwatt has obtained the partial fractions of

$$(\sum_{k=0}^{k=n} m_k x^{n-k})/(x + a)^p,\ n \geqq p,$$

and

$$(\sum_{k=0}^{k=n} m_k x^{n-k})/(x^2 + ax + b)^p,\ n \geqq 2p.$$

The coefficients of the first are expressed as single summations and those of the second as double summations. The literature on the subject shows that heretofore the coefficients have been expressed as determinants.

30. Professor I. J. Schwatt: *Expansion of powers of infinite series.*

If the series to be expanded to the pth power is not summable, the coefficient of the general term of the expansion is the product of $p - 1$ summations. If the series is summable the work is reduced to the evaluation of the general derivative

of a function of a function for the vanishing primary variable. The methods employed are believed to be new.

31. Professor Olive C. Hazlett: *A symbolic theory of formal modular invariants.*

At the meeting in connection with the Chicago Colloquium, Professor Hazlett introduced a symbolic notation for the formal modular invariants of a binary form f of degree m with respect to the general Galois field $GF[p^n]$ of order p^n. To this end, the form f was expressed as the product of m symbolically distinct factors $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$, $\beta_x = \beta_1 x_1 + \beta_2 x_2$, $\cdots$. It was there shown that all formal modular invariants of f are expressible as polynomials in a finite number of symbolic expressions which behave like invariants. These invariant symbols include determinantal symbols of the type

$$(\alpha\beta) = \alpha_1\beta_2 - \alpha_2\beta_1,$$

which arise in the theory of ordinary invariants of f, and more complicated ones which are peculiar to the theory of modular invariants. In the present paper, it is shown that, if a formal modular invariant I is isobaric, it is necessarily congruent (modulo p) to a polynomial in determinantal symbols of type $(\alpha\beta)$ and hence is an ordinary invariant of f, though not necessarily a rational one. In case I is not isobaric, then it is congruent (modulo p) to an ordinary invariant (rational or irrational) of f and certain related forms. Similar theorems hold for formal modular covariants of f.

32. Dr. Norbert Wiener and Dr. J. L. Walsh: *The equivalence of expansions in orthogonal functions.*

The authors develop necessary and sufficient conditions for the identity of the convergence properties of different expansions in orthogonal functions. They apply these notions to various related problems. The chief theorem is that two closed sets of normal orthogonal functions $\{\varphi_n\}$ and $\{\psi_n\}$ have identical convergence properties when and only when the φ-series for the ψ's converge uniformly, while

$$\int_0^1 \{\sum_1^n [\varphi_n(x)\varphi_n(y) - \psi_n(x)\psi_n(y)]\}^2 dy$$

is uniformly bounded in n and x.

ARNOLD DRESDEN,
Secretary of Chicago Section.

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Secretary.

THE MOMENT OF INERTIA IN THE PROBLEM OF N BODIES*

BY PROFESSOR W. D. MACMILLAN

The differential equations of the problem of n bodies are

$$m_i x_i'' = \frac{\partial V}{\partial x_i}, \; m_i y_i'' = \frac{\partial V}{\partial y_i}, \; m_i z_i'' = \frac{\partial V}{\partial z_i}, \; (i = 1, \cdots, n)$$

in which the potential function is

$$V = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k^2 \frac{m_i m_j}{r_{ij}}, \; (j \neq i).$$

The kinetic energy of the system is

$$T = \tfrac{1}{2} \sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^2).$$

The moment of inertia of the n bodies considered as point masses with respect to the origin is

$$I = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2 + z_i^2).$$

On differentiating this expression for the moment of inertia twice with respect to the time, we find

$$\tfrac{1}{2} I'' = \sum_{i=1}^{n} m_i (x_i x_i'' + y_i y_i'' + z_i z_i'') + \sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^2).$$

The last term of this expression is twice the kinetic energy of the system, $2T$. The first term of the right member, in virtue of the differential equations of motion, may be written in the form

$$\sum_{i=1}^{n} \left(x_i \frac{\partial V}{\partial x_i} + y_i \frac{\partial V}{\partial y_i} + z_i \frac{\partial V}{\partial z_i} \right).$$

Since V is a homogeneous function of degree -1 of the quantities x_i, y_i, z_i, we have

$$\sum_{i=1}^{n} \left(x_i \frac{\partial V}{\partial x_i} + y_i \frac{\partial V}{\partial y_i} + z_i \frac{\partial V}{\partial z_i} \right) = -V.$$

Consequently the differential equation for the moment of

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inertia may be written in the form

$$\tfrac{1}{2}I'' = -V + 2T.$$

Since $T - V = \mathcal{E}$ is the total energy of the system, and is therefore constant, we can eliminate the kinetic energy from the preceding equation, and we obtain

$$\tfrac{1}{2}I'' = V + 2\mathcal{E}.$$

This equation was derived by Jacobi and discussed by him in considerable detail in his *Vorlesungen über Dynamik*.

Since the moment of inertia is homogeneous of degree + 2 with respect to the quantities x_i, y_i, and z_i, and since V is homogeneous of degree − 1, it follows that the product

$$VI^{1/2} = C$$

is a homogeneous function of degree zero, which in certain cases is an actual constant with respect to the time. In any event, C is a function which is always positive and which depends only upon the masses and their relative distributions, if we choose the center of mass as the origin, and not at all upon the size of the configuration. Eliminating V from the differential equation by the introduction of the function C, we have

$$I'' = 2CI^{-1/2} + 4\mathcal{E}.$$

As Jacobi pointed out, this equation shows that if $\mathcal{E}$ is a positive constant at least one of the bodies must recede to infinity, and therefore if the system is to be a permanent one, $\mathcal{E}$ must be negative. To exhibit this fact, we shall take $\mathcal{E} = -E$, and we shall suppose that we are dealing with a permanent system, so that E is positive. Then

$$I'' = 2CI^{-1/2} - 4E.$$

If C is a constant, this equation can be integrated. The first integral is

$$I^{1/2} = \frac{2}{E}[C_1^2 - (C - 2EI^{1/2})^2],$$

the constant of integration being taken to be $2[C_1^2 - C^2]E^{-1}$. Evidently the new constant C_1^2 cannot be negative. If it is zero, $I^{1/2}$ has the constant value $I^{1/2} = C/(2E)$.

The second integral is

$$\frac{C}{(2E)^{3/2}} \sin^{-1}\left(\frac{2EI^{1/2} - C}{C_1}\right)$$

$$- \frac{1}{(2E)^{3/2}} \sqrt{C_1^2 - (C - 2EI^{1/2})^2} = t - t_0,$$

which shows that $I^{1/2}$ is a periodic function of t with the period

$$P = \frac{2\pi C}{(2E)^{3/2}},$$

and that it oscillates about the mean value $C/(2E)$, the amplitude of the oscillation having the value C_1. It is worthy of note that the period of the oscillation is independent of the amplitude. It depends only upon the energy of the system and upon the nature of the configuration.*

In the problem of two bodies, the function C is necessarily a constant and has the value $(m_1m_2)^{3/2}/\sqrt{m_1 + m_2}$. In the problem of three bodies it is a constant for the equilateral triangular solution and for the straight line solution of Lagrange. It is constant for Longley's parallelogram solution of the problem of four bodies, and for Moulton's straight line solution of the problem of n bodies. In short it is constant for every solution of the problem of n bodies for which a definite geometric configuration is preserved throughout the motion, i.e., one in which the ratios of the mutual distances are constants.

This condition is sufficient, but there is nothing to indicate that it is necessary. In the globular star clusters we have a

* If we replace $I^{1/2}$ in this integral by the radius of gyration, ρ, by means of the relation $I^{1/2} = \rho\sqrt{M}$ where $M = \Sigma m_i$; and make the further substitutions

$$\rho = \frac{C - C_1 \cos\varphi}{2E\sqrt{M}}, \qquad \frac{(2E)^{3/2}}{C}(t - t_0) = \tau - \pi/2, \qquad \frac{C_1}{C} = e,$$

it reduces to Kepler's Equation $\varphi - e \sin\varphi = \tau$, with

$$\rho = \frac{C}{2E\sqrt{M}}(1 - e\cos\varphi).$$

These equations are familiar in the problem of two bodies.

type of organization in which a steady state has apparently been reached. In such an aggregation, the function C must be very nearly constant over extraordinarily long intervals of time. If our galaxy is such a star cluster, it is oblate rather than globular. Assuming that it is composed of 1.5×10^9 stars such as our sun with an equatorial radius of 2000 parsecs (6,600 light years) and a polar radius of 600 parsecs, and that the mean velocity of the stars is 25 kilometers per second, the period of oscillation (if any exists) is 25,600,000 years and the period of a circular orbit about its equator is 74,300,000 years. If we assume a stellar density only one-fifth as great and a stellar velocity of 40 kilometers per second instead of 25 kilometers per second, the period of oscillation is 97,000,000 years, and the period of the circular orbit about the equator is 166,000,000 years. It should be added, however, that even though the function C is a constant (or nearly constant) for the galaxy, it is not necessary that there should be any oscillation. The moment of inertia may be a constant.

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SUBSTITUTIONS COMMUTATIVE WITH EVERY SUBSTITUTION OF AN INTRANSITIVE GROUP*

BY G. A. MILLER

It is well known how to find all the substitutions which are separately commutative with every substitution of a transitive group G of degree n and which involve no letter except such as are found in G. If G_1 is composed of all the substitutions of G which omit a given letter of G, and if the degree of G_1 is $n - \alpha$, then all the substitutions on the letters of G which are separately commutative with every substitution of G constitute a subgroup K of order α, and all the substitutions of K besides the identity are regular and of degree n.

When G is intransitive, the results are not quite so elegant, but as they are often useful it seems desirable to state them

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explicitly. It is evident that the direct product of the groups composed separately of the substitutions on the letters of the various transitive constituents of G which are commutative with all the substitutions of these constituents is a group K_0 composed of substitutions which are separately commutative with every substitution of G. A necessary and sufficient condition that K_0 include all the substitutions on the letters of G which are commutative with every substitution of G, is that G have no two transitive constituents which are simply isomorphic with each other and in which the subgroups composed of all the substitutions which omit a letter correspond to such sub-groups. This results directly from the following theorem:

THEOREM. *A necessary and sufficient condition that there exist a substitution which interchanges h systems of intransitivity and is commutative with every substitution of the intransitive group involving these systems, is that there be a simple isomorphism between the h transitive groups which constitute these systems, such that a subgroup composed of all the substitutions which omit a letter of one of these h transitive groups corresponds to such a subgroup of the other.*

This theorem results almost directly from the fact that every automorphism of a transitive group in which a subgroup composed of all the substitutions which omit a given letter corresponds to such a subgroup, can be obtained by transforming the transitive group by substitutions involving only the letters of this group. In fact, it is evident that we can establish an automorphism of a transitive group in such a way that a substitution t on the letters of the two transitive constituents thus obtained interchanges these constituents and is commutative with every substitution of the intransitive group formed by them. By transforming one of these constituents by a substitution on its own letters which transforms it into itself the isomorphism mentioned in the theorem is obtained, and t is transformed thereby into a substitution which is commutative with every substitution of the intransitive group obtained in this way.

If G involves h transitive constituents which can be trans-

formed transitively by means of substitutions which are commutative with every substitution of the group formed by these constituents, then these constituents can be transformed into each other according to the symmetric group of degree h. The total number of such substitutions on the letters of the h constituents can therefore be obtained by extending the part of the direct product K_0 which relates to these constituents by means of the substitutions of this symmetric group of degree h. As a very special case of this theorem we have that all the substitutions which are commutative with every substitution of the intransitive group obtained by establishing a simple isomorphism between n symmetric groups of degree $n > 2$ and $n \neq 6$ constitute a group which is a conjugate of this intransitive group.*

From the preceding developments it results that all the substitutions which are commutative with every substitution of an intransitive group can be found by first constructing K_0 and then extending this K_0 successively by means of the symmetric groups on h transitive constituents if any exist, where h is always taken as large as possible. If one such h is not equal to the number of transitive constituents involved in G, then the group composed of all the substitutions which are commutative with every substitution of G is a direct product of the groups relating to the constituents which are transitively connected. While the order of each of the constituents of G which can be transitively connected must be the same, those which can not be thus connected may evidently have different orders.

THE UNIVERSITY OF ILLINOIS

* Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 41. It may be noted that the condition $n \neq 6$ should have been added there since the symmetric group of degree 6 admits outer isomorphisms.

SUMMABLE INFINITE DETERMINANTS*

BY WILLIAM L. HART

1. *Introduction.* It has been customary to define the value of the infinite determinant

$$D = |a_{ij}|_{i,j=1,2,\cdots} \tag{1}$$

by the equation

$$D = \lim_{n\to\infty} D_n \qquad (D_n = |a_{ij}|_{i,j=1,2,\cdots,n}), \tag{2}$$

in case the limit exists. Supposing that the a_{ij} are any complex quantities, we may state

DEFINITION. *The determinant* (1) *is summable to the value* D *in case the sequence* (D_n) *is summable to* D.

In this definition the term *summable* may be understood in any one of the various senses which have been given† to that term. It is easily seen that if D is summable, its value remains unchanged if we interchange its rows and columns. The sign of D is changed if two contiguous rows or columns are interchanged. If all the elements of a row, or column, are multiplied by a constant K, the new determinant obtained has the value KD.

It is the purpose of this note to present certain types of infinite determinants which are summable in the sense of the Cesàro first mean and, in the future, the unqualified word *summable* will be applied to determinants of this variety. In § 2, there will be given a proof of the summability and a discussion of certain properties of a class of determinants analogous to the Von Koch normal‡ infinite determinants. In § 3 the summability of a somewhat different type of determinants will be established.

2. *A Certain Class of Determinants.* A simple example of an infinite determinant which is summable to the value 1/2 is

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† E. Borel, *Les Séries divergents*, p. 87; Hurwitz, Report on divergent series, this BULLETIN, vol. 28, 1922, pp. 17–36.

‡ Cf. Kowalewski, *Determinantentheorie*, p. 372.

$$(3) \qquad D = \begin{vmatrix} 0 & 1 & & & & \\ -1 & \pm 1 & & & & \\ & & 0 & 1 & & \\ & & -1 & \pm 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & \cdot \end{vmatrix},$$

where all elements are zero except for those in the main diagonal minors of order 2, which are displayed, and where the $\pm$ signs in the main diagonal may be selected at will.

For (3), we find that $D_{2k} = 1$ and $D_{2k-1} = 0$ for $k = 1, 2, \cdots$. The determinant (3) would remain summable to the value 1/2 if arbitrary elements b_{ij} were substituted for the zero elements situated above the elements $+1$, in the columns of even index, and to the left of the elements -1, in the rows of even index.

Let the arbitrary quantities b_{ij} of the preceding paragraph be chosen as either $+1$, -1, or 0, and, after this selection, let the element of (3) in row i and column j be denoted by e_{ij}.

THEOREM I. *In* (1) *suppose that* $a_{ij} = e_{ij} + c_{ij}$, *where* $\Sigma_{ij=1}^{\infty} |c_{ij}|$ *converges. Then the determinant* D *of* (1) *is summable and*

$$(4) \qquad 2D = D_2 + (D_4 - D_2) + \cdots + (D_{2h} - D_{2h-2}) + \cdots.$$

First let us consider certain properties of the infinite matrix

$$A = (c_{ij})_{ij=1, 2, \ldots}.$$

By a minor of A of order p we shall mean a determinant of order p formed by those elements of a certain set of p rows of A which are located in a certain set of p columns. Let I represent the sum of the absolute values of all terms which enter in the expansions of all minors, of all orders, of the matrix A. Then it is well known that I converges.*

In order to establish the theorem, we note that the determinants D_n in the present case have the following properties when expanded in terms of the quantities c_{ij}:

(*a*) For every n, D_n is equal to the sum of certain terms from

* Cf. Kowalewski, loc. cit., p. 374.

I, with positive or negative signs attached, plus 1, if n is even, and plus zero, if n is an odd integer.

(b) For every value of $h > 1$, D_{2h} contains all terms of D_{2h-2}, and, moreover, the new terms occurring in D_{2h} do not occur in any D_{2k} if $k < h$.

(c) For every value of h, D_{2h-1} consists of terms which are not found, even with a change of sign, in any other determinant D_{2k-1} $(k \neq h)$.

The property (a) results from the fact that the elements e_{ij} are ± 1 or 0. One verifies (b) by expanding D_{2h} according to the elements

$$c_{1,\,2h-1},\ c_{2,\,2h-1},\ \cdots,\ c_{2h-1,\,2h-1},\ c_{2h,\,2h-1} - 1, \tag{5}$$

in its $(2h-1)$-th column. We establish (c) by expanding D_{2h-1} according to the elements of its last column, which consists of all the quantities (5) except the last.

To establish the summability of D, we must show the existence of $\lim_{n\to\infty} S_n$ where

$$S_n = \frac{D_1 + \cdots + D_n}{n}. \tag{6}$$

When $n = 2k$ let us write

$$S_{2k} = \frac{R_{2k}}{2k} + \frac{T_{2k}}{2k},$$

where

$$R_{2k} = D_1 + D_3 + \cdots + D_{2k-1}$$

and

$$T_{2k} = D_2 + D_4 + \cdots + D_{2k}.$$

In view of properties (a) and (c) above it follows that $|R_{2k}| \leqq I$. Consequently,

$$\lim_{k\to\infty} \frac{R_{2k}}{2k} = 0.$$

It follows from properties (a) and (b) that each bracket in the series

$$K = D_2 + \cdots + (D_{2h} - D_{2h-2}) + \cdots,$$

when written in terms of the quantities c_{ij}, consists of terms from I with various signs attached. Since no term occurs in

more than one bracket, it is seen that K converges. Therefore,

$$\lim_{k\to\infty}\frac{T_{2k}}{k}=K=\lim_{k\to\infty}D_{2k},$$

and it follows that $\lim_{k\to\infty}S_{2k}=K/2$. As a consequence of the simple identity

$$S_{2k+1}=\frac{R_{2k+2}}{2k+1}+\frac{T_{2k}}{2k}\frac{2k}{2k+1},$$

it is verified that $\lim_{n\to\infty}S_n=K/2$, which completes the proof of the theorem.

It is evident that the algebraic complement* A_{ij} of each element a_{ij} in the determinant of Theorem I is itself a determinant of the same type. For the complement is obtained by replacing a_{ij} by $+1$, or -1, as the case may be, and by substituting zero for all other elements in the ith row or in the jth column.

If we consider the determinant D' resulting from (3) by the addition of c_{ij} to the element in the ith row and jth column, for every i and j, it follows from Theorem I that D' is summable. Let E represent a determinant obtained from D' by replacing a finite number of its rows (or of its columns) by a bounded set of numbers. Then we could prove that E is summable by the method used in establishing Theorem I.

Let us consider the sequence $(t_n;\ n=1, 2, \cdots)$ defined by

$$t_{2k-1}=2k, \qquad t_{2k}=2k-1 \quad (k=1, 2, \cdots), \tag{7}$$

as the normal order for the arrangement of the positive integers. Then a sequence $p=(j_h;\ h=1, 2, \cdots)$, containing all the positive integers, will be called an *alteration* of the order (7) if, for all values of h sufficiently large, $j_h=t_h$. Thus, for every alteration p, there exists a smallest even index $h=2m$ such that $j_k=t_k$ if $k\geqq 2m-1$. Consider the first $(2m-2)$ terms of p,

$$j_1, \cdots, j_{2m-2}, \tag{8}$$

which give a permutation of the numbers $(1, 2, \cdots, 2m-2)$. Let us call p an *even* or *odd* alteration according as (8) is obtainable from the order $(1, \cdots, 2m-2)$ by an even or by

* Cf. Kowalewski, loc. cit., pp. 378, 382.

an odd number of successive interchanges of neighboring numbers. It is well known* that the evenness or oddness of p is independent of the way in which these interchanges were made.

To each alteration p let us make correspond the term

$$P_p = \pm \prod_{k=1}^{2m-2} a_{k,\, j_k} [\prod_{k=m}^{\infty} (- a_{2k-1,2k}\, a_{2k,\, 2k-1})],$$

where the $+$ sign is selected if p is even and the minus if p is odd. Suppose that (1) is a determinant of the type E mentioned above. Then it can be proved that

$$2E = \sum_{(all\ p)} P_p, \tag{9}$$

where the summation is extended over all alterations p of the system of positive integers. On the basis of this result, it can then be shown that, for every j and for every k,

$$E = \sum_{i=1}^{\infty} a_{ij} A_{ij} = \sum_{j=1}^{\infty} a_{kj} A_{kj}, \tag{10}$$

where A_{ij} is the algebraic complement of the element a_{ij} in (1). The proofs of (9) and (10) will not be given since the reasoning would be practically identical with that used in the derivation of similar results in the theory of normal infinite determinants.† By use of (9) and (10), it would be possible to develop a theory for the solution of the infinite system of equations

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i, \qquad (i = 1, 2, \cdots), \tag{11}$$

provided that the determinant (1) formed by the coefficients a_{ij} is of the type D' defined above. A very casual inspection, however, shows that system (11) could easily be transformed into a system of a well known type whose solution could be obtained in terms of normal infinite determinants.

The determinants we have considered were obtained from (3) by simple transformations. If we should take more complicated determinants than (3) as our points of departure we could obtain results similar to those derived above. For example, consider the determinant of order q given by

* Cf. Kowalewski, loc. cit., p. 9.

† Cf. Kowalewski, loc. cit., §§ 154, 155.

$$(12)\qquad \begin{vmatrix} 1 & & & & \\ & \ddots & & & \\ & & 10 & \cdots \pm 1 & (p\text{th row}), \\ & & 10 & & \\ & & \ddots & & \\ & & & 10 & (q\text{th row}) \end{vmatrix}$$

where all elements not explicitly indicated are zero and where the ambiguous sign in the last column is selected in such a way that the determinant has the value $+1$. In the infinite determinant D of (1) let the elements a_{ij} in the rows of index $mq+1, mq+2, \cdots, mq+q$, for all values $m = 1, 2, \cdots$, be selected as zero, except for those in the columns $mq+1$, $mq+2, \cdots, mq+q$. Choose these q^2 elements as those of the determinant (12). It is easily verified that D is summable to the value p/q. With this determinant D as a starting point instead of the determinant (3), the same methods that were used above suffice to establish a theorem analogous to Theorem I.

The results stated after Theorem I would also have their analogies in the present case. Once more it should be noted that the determinants considered in this paragraph would not enable one to solve any problem in connection with infinite systems of linear equations which could not equally well be solved by the known theory of normal infinite determinants.

3. *Another Type.* An example of the type* of determinants to be considered in the present section is given by (1) in case

$$(13)\qquad a_{ij} = 0, \quad (i \neq j); \qquad a_{ii} = (-1)^i, \quad (i = 1, 2, \cdots).$$

Then D is summable to the value zero and satisfies the equation

$$(14)\qquad D = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1} + \lim_{n\to\infty} D_{2n}).$$

Another example of this type is given by (1) if

$$(15)\qquad \begin{aligned} &a_{ij} = 0, \qquad (i \neq j); \\ &a_{2i-1,\,2i-1} = \alpha, \qquad (a_{2i,\,2i} = \beta, i = 1, 2, \cdots; \alpha\beta = 1). \end{aligned}$$

In this case D satisfies (14), where the first limit is α and the second is 1, so that $D = (\alpha+1)/2$. It is obvious that (15) may be generalized to the case where the main diagonal

* The author acknowledges his indebtedness to Professor L. L. Silverman for suggesting the consideration of the determinants treated in this section.

elements are $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_1, \cdots, \alpha_k, \cdots$ and where all other elements are zero, under the supposition that

$$\alpha_1 \alpha_2 \cdots \alpha_k = 1. \tag{16}$$

If, under conditions (13), we should add c_{ij} to the element a_{ij} of D, for all (i, j), the resulting determinant would be summable and its value would be given by (14) in case the (c_{ij}) satisfied the condition of Theorem I. The proof of this statement would be almost a repetition of that of Theorem I.

Let D' represent the determinant resulting from (1), under conditions (15), by the addition of c_{ij} to a_{ij}.

THEOREM II. *In the sum I of Theorem* I *let* p_k *represent the sum of all terms containing exactly* k *factors* c_{ij}. *In* (15) *suppose that* $|\alpha| > 1$ *and that*

$$\sum_{k=1}^{\infty} p_k |\alpha|^k \tag{17}$$

converges. Then, D' *is summable and satisfies the relation*

$$D' = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1}' + \lim_{n\to\infty} D_{2n}'). \tag{18}$$

The properties of summable sequences make it clear that the theorem will be completely established if we can show that the two limits in (18) exist. Let us consider the convergence of the sequence (D_{2n+1}'). The last two rows of D_{2n+1}' are:

$$\begin{matrix} c_{2n,1}, & c_{2n,2}, & \cdots, & c_{2n,2n} + \beta, & c_{2n,2n+1} \\ c_{2n+1,1}, & c_{2n+1,2}, & \cdots, & c_{2n+1,2n}, & c_{2n+1,2n+1} + \alpha. \end{matrix} \tag{19}$$

In expanding D_{2n+1}' by Laplace's Rule,* according to the minors of the last two rows, it is verified that we obtain

$$D_{2n+1}' = D_{2n-1}' + P_{2n+1}, \tag{20}$$

where P_{2n+1} consists of terms containing α, β and factors c_{ij}. Let the expression *frame of a term* of P_{2n+1} refer to the absolute value of the product of those factors c_{ij} entering in the term. The expression P_{2n+1} has the following properties:

(*a*) All terms contain at least one c_{ij} from those in (19) and hence the frames of terms of P_{2n+1} are distinct from those of P_{2h+1} for $h < n$.

(*b*) There are no repetitions among the frames of the terms of P_{2n+1}.

* Cf. Bôcher, *Introduction to Higher Algebra*, p. 24.

(*c*) After the relation $\alpha\beta = 1$ has been used, the power of α entering in a term of P_{2n+1} is at most one unit greater than the number of factors c_{ij} in that term. If a power of β remains after use of $\alpha\beta = 1$, the absolute value of the term is increased by neglecting the power of β because $|\beta| < 1$.

It is seen that the convergence of the sequence (D_{2n+1}') is equivalent to that of the series

$$D_1' + P_3 + P_5 + \cdots + P_{2n+1} + \cdots. \tag{21}$$

Let $\bar{P}_{2n+1}$ represent P_{2n+1} with all of its terms replaced by their absolute values. As a consequence of the properties (*a*), (*b*) and (*c*) it follows that the convergence of (17) implies that of (21) because $\bar{P}_3 + \bar{P}_5 + \cdots \leqq |\alpha| \sum_{k=1}^{\infty} |\alpha|^k p_k$. Hence the sequence (D_{2n+1}') converges. A proof of the same nature as that just given would establish the convergence of the sequence (D_{2n}'). Therefore, Theorem II may be considered completely proved. If we had supposed $|\beta| > 1$, an analogous proof would have shown D' to be summable under the same assumption as was made in Theorem II.

Let E represent the determinant associated with (16) after c_{ij} has been added to the element in row i, column j, for all (i, j). The determinant E is summable if condition (17) is satisfied, where we understand $|\alpha|$ to represent the absolute value of the product of all factors α_h from (16) which satisfy $|\alpha_h| > 1$. The proof of this statement would be similar to that given for Theorem II. E would satisfy the equation

$$E = \frac{1}{k}\left(\lim_{n\to\infty} E_{kn} + \lim_{n\to\infty} E_{kn+1} + \cdots + \lim_{n\to\infty} E_{kn+k-1}\right). \tag{22}$$

Certain interesting questions regarding summable determinants remain unanswered in this note. It may be that Theorem II and its generalization, in connection with determinant E, are true under hypotheses less restrictive than those of this paper. More generally, it would be of interest to know whether a determinant (1), which is summable, remains so if quantities (c_{ij}) are added to its elements, where the c_{ij} satisfy the condition of Theorem I.

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ALGEBRAIC GUIDES TO TRANSCENDENTAL PROBLEMS.*

BY R. D. CARMICHAEL

1. *Introduction and Historical Sketch.* Up to the present time algebraic guides to transcendental problems have been employed extensively through only a small part of the range of subject matter to which they are so adapted as to yield important characteristic values in suggested theorems and in methods of proving them. This interaction between algebraic and transcendental analysis has attracted greater attention in the theory of integral equations than elsewhere. The relation between the theory of integral and of algebraic equations seems to have been first noticed by Volterra, who pointed out (TORINO ATTI, 1896, pp. 311–323) that a Volterra integral equation of the first kind may be regarded as in a certain sense a limiting form of a system of n linear algebraic equations in n variables as n becomes infinite. It is clear from Volterra's remarks in 1896 that the same is true of the Volterra equation of the second kind, though this fact was not then mentioned explicitly. In 1913 in his *Leçons sur les Équations Intégrales et les Équations Intégro-différentielles*, Volterra brings out in detail (pp. 30–33, 40–52) the connection between the algebraic theory and his equation of the second kind, and less fully (pp. 56 ff.) the connection between the algebraic theory and his equation of the first kind. He indicates (pp. 71 ff.) extensions of the method to systems of integral equations and to equations and systems with multiple integrals, and also (pp. 138 ff.) to the theory of permutable functions. (See also the preface and pp. 33, 102, 117 for remarks on the history of the subject and for references.) We shall set forth the character of the method by a brief indication of the nature of Volterra's treatment of the equation of the second kind.

* Address as retiring Chairman of the Chicago Section of this Society, Toronto, December 27, 1921. Read for the author by Professor Arnold Dresden.

If in the equation

$$\varphi(x) = u(x) + \int_0^x K(x, \xi)u(\xi)d\xi$$

we replace the integral by the limit to which it is equal by definition, we have the relation

$$\varphi(x) = u(x) + \lim_{n\to\infty} \sum_{i=1}^{n} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1}),$$

where the points $\xi_0 = 0, \xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n = x$ of the interval $(0, x)$ are so distributed that the greatest difference $\xi_i - \xi_{i-1}$ approaches zero with $1/n$. Approximating to the latter equation is the following:

$$\varphi(x) = u(x) + \sum_{i=1}^{n-1} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1});$$

and in particular the system

$$\varphi(\xi_s) = u(\xi_s) + \sum_{i=1}^{s-1} a_{si}u(\xi_i), \quad (s = 1, 2, \cdots, n),$$

where $a_{si} = K(\xi_s, \xi_i)(\xi_i - \xi_{i-1})$. Now this algebraic system has a solution in the form

$$u(\xi_s) = \varphi(\xi_s) + \sum_{i=1}^{s-1} c_{si}\varphi(\xi_i), \quad (s = 1, 2, \cdots, n);$$

and reciprocally the former system affords the solution of the latter for the $\varphi(\xi_i)$ in terms of the $u(\xi_i)$. By analyzing these two systems Volterra obtains several fundamental relations between them. Then he proceeds heuristically to the limiting forms of these relations after the manner of forming an integral as the limit of a sum, thus obtaining the main principles upon which rests his solution of the given integral equation—these principles then being established *de novo* by methods suggested by the algebraic analysis.

The methods of Volterra may be extended to the case of the Fredholm equation; but an added difficulty arises in the new situation from the fact that the basic algebraic system has now a general determinant depending upon the kernel K rather than the much simpler determinants of the Volterra treatment. It was Fredholm's achievement to see how the method of Volterra could be extended so as to pass from the solution of a system of linear algebraic equations to the solution of the Fredholm integral equation of the second kind. Again

in the case of Fredholm, the method was used merely as a heuristic guide for discovering the facts and suitable methods for their proof. With Hilbert arose for integral equations a marked extension of the method through his plan of deducing the results by limiting processes from algebraic propositions as the number of variables becomes infinite.

But the better-known applications to integral equations do not afford the earliest important uses of algebraic guides to transcendental problems. A limiting process having certain points in common with that of Volterra was employed by Cauchy in his proof of the existence of integrals of a system of differential equations, and has been preserved in the lectures of Moigno published in 1844. Considerably simplified and clarified by Lipschitz it stands today under the name of the Cauchy-Lipschitz method (with the extensions of Picard and Painlevé) as one of the principal means for establishing the existence of integrals of differential equations. (See, for instance, Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 2, pp. 68–74.)

In some respects a closer but still a rather remote analogy with the work of Volterra is afforded by the unpublished method by which Sturm was led to many of the results of his great memoirs of 1836. On page 186 of the first volume of LIOUVILLE'S JOURNAL, Sturm tells us that it was from the solutions of the difference equation

$$L_i u_{i+1} + M_i u_i + N_i u_{i-1} = 0, \quad (i = 0, 1, 2, \cdots),$$

which is nothing more than a somewhat disguised special form of a system of linear algebraic equations, that he was led to the subject of his first memoir of 1836, the memoir in which his more characteristic results for differential equations are to be found; and that the method by which the latter were obtained from the former was one of passing by a limiting process from finite to infinitely small differences. He adds that he had also found for the difference equation properties which are not susceptible of being carried over to differential equations. From these remarks, which he seems never to have elaborated anywhere in his published writings, it is clear that Sturm was

led to many of his classic results by a fundamental algebraic guide to the transcendental problems treated.

If one has in mind the suggestive remark of Sturm and the limiting process of Cauchy's existence proof, there is no longer any difficulty in carrying through the limiting process from the difference to the differential equation and of obtaining properties of the solution of the latter directly from those of the former. This was done in an interesting case by Porter in 1902 (ANNALS OF MATHEMATICS (2), vol. 3 (1902), pp. 55–70) more than two years before Hilbert in 1904 took a similar step for integral equations. If we remember that the difference equations used in this analysis are but condensed forms of certain systems of linear algebraic equations, we shall see that the processes which we have usually associated principally with the theory of integral equations were used heuristically in the theory of differential equations long before they were similarly employed for integral equations, and that they were used some years earlier in the former than in the latter in the way of a rigorous passage to the limit.

In Bôcher's Fifth International Congress address and in his *Les Méthodes de Sturm* (see especially chapter 2, pp. 14–42) there is brought out more clearly and more fully than by any of his predecessors the intimate relation that exists between the theory of systems of linear algebraic equations and differential equations, particularly with boundary conditions. Several important theorems for differential systems are obtained as immediate analogues of well known fundamental theorems for algebraic systems, and several intimately related aspects of the two problems are brought out forcibly by the statement of theorems common to the two and identical in their principal characteristics.

It will be observed that all the problems treated in these cases (with the exception of that of Cauchy) are linear in character. This is not accidental; it arises from two facts. Many of the profound phenomena of nature are subject to laws whose expression in mathematical form gives rise to fundamental linear problems of several kinds. Logically the

simplest and historically the first to be treated in detail of the linear problems of pure mathematics are those having to do with systems of linear algebraic equations. And these hold the place of greatest importance, both on account of their simplicity and relatively complete development and on account of their suggestiveness in leading the way to transcendental linear problems which emerge from a direct consideration of the natural limiting cases of algebraic systems under the guidance of current problems of transcendental analysis.

My principal purpose in this address is to discuss two types of algebraic theorems and the transcendental problems to whose solution they lead the way. In the one class we shall have oscillation and comparison theorems, and in the other theorems of expansion in orthogonal functions and their generalizations. To one acquainted with the relevant theory of differential and integral equations, it is clear that there is already at hand a large body of doctrine having to do with certain transcendental problems in each of these domains, namely, the classic oscillation and comparison theorems of Sturm for differential equations of the second order, the Sturm-Liouville expansions, expansions by means of the biorthogonal functions arising from Birkhoff's theory of differential systems and their adjoints, and the expansion theorems arising in the theory of integral equations.

The principal algebraic theorems to which I shall direct your attention were conceived in the first place by considering what properties of certain approximating algebraic systems correspond to the properties already established for the transcendental problems which have been treated. It was not difficult to arrive at the corresponding theorems for the special algebraic systems which were involved in certain of these cases; and the theorems once in hand for the special systems were readily extended to fairly general classes of algebraic systems. With this much accomplished, one is in possession of algebraic facts suitable to serve as a guide to a large class of transcendental problems having certain analogies with the problems which suggested the algebraic theorems in the first

place. The interaction thus set up between the algebraic and the transcendental problems has a useful power of leading forward to the discovery of important results.

In order to have clearly in mind the nature of the connection between a differential equation and a system of algebraic equations, we shall set up the relation in one of the simplest important cases. A differential equation, with or without boundary conditions, may be realized in an infinite number of ways as the limiting form of an algebraic system, so that there is always room for choice in setting up the system, and in fact need for care that it shall be done in a convenient way. In connection with the equation of second order

$$u''(x) + \varphi(x)u(x) = 0,$$

it is often convenient to employ the approximating equation

$$\frac{u(x+2\delta) - 2u(x+\delta) + u(x)}{\delta^2} + \varphi(x+\delta)u(x+\delta) = 0,$$

which reduces to the differential equation when δ approaches zero, provided that u and φ are subject to appropriate conditions. This equation reduces to

$$u(x) + \{\delta^2\varphi(x+\delta) - 2\}u(x+\delta) + u(x+2\delta) = 0.$$

If we are concerned with the original differential equation when x ranges over an interval (ab), we may take $\delta = (b-a)/n$ where n is an integer. Then, giving to x in the last equation the values a, $a+\delta$, $a+2\delta$, $\cdots$, $a+(n-2)\delta$, we have an algebraic system of $n-1$ equations of the form

$$u(a+i\delta) + a_i u(a+\overline{i+1}\,\delta) + u(a+\overline{i+2}\,\delta) = 0, \qquad (i = 0, 1, \cdots, n-2),$$

from which to find the $n+1$ unknown quantities $u(a)$, $u(a+\delta)$, $\cdots$, $u(a+n\delta)$. From properties of the solutions of this algebraic system one can pass back heuristically to properties of the solutions of the differential equation and then can establish these properties by a *de novo* argument suggested by the methods for dealing with the algebraic system. This special case may suggest certain principal characteristics of the general method as applied to differential equations.

The two instances given, the one just developed and that of Volterra, do not by themselves afford an adequate suggestion of the range of applicability of the method. From these cases one can see in part how it works for integral equations and ordinary differential equations. In a similar way it may be brought to bear upon the theory of difference and q-difference equations, both ordinary and partial, and the theory of partial differential equations. Moreover, if we pass from any of these cases by another limiting process of Volterra to such limiting forms as his linear integro-differential equations, or to linear integro-difference and integro-q-difference equations, or to various linear systems combining the properties of these mentioned types of transcendental equations, we shall be able to look upon any one of these directly as a limiting case of an algebraic system under some appropriate method of passing to the limit. In fact, it is probable that certain essential elements of these algebraic guides to transcendental problems can be realized in the case of any transcendental linear problem to which one may be led naturally.

2. *Algebraic Oscillation and Comparison Theorems.* Let us consider a graphical representation of the set of real constants $u_1, u_2, \cdots, u_n$ obtained in the following manner (cf. M. B. Porter, ANNALS OF MATHEMATICS (2), vol. 3 (1901), p. 56). On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars, two of which are at the ends of the segment, while the other $n - 2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coordinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. On the ith axis let us take a point at a distance $|u_i|$ from the original segment, and above it or below it according as u_i is positive or negative. Having done this for each value i of the set $1, 2, \cdots, n$, join by straight line segments the point on each axis but the last to the point on the adjacent axis to the right. We thus obtain a broken line

which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions, or of the set of constants $u_1, u_2, \cdots, u_n$.

This broken line may be taken also as the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ has been obtained from the set of constants $u_1, u_2, \cdots, u_n$ by linear interpolation with respect to the given n axes. The points at which (values of s for which) this broken line cuts the original line segment (viewed as the axis of s) we shall call indifferently the zeros of the set of constants or of the function $u(s)$.

Let us now suppose that we have a given real-valued single-valued function $v(s)$ of the real variable s, continuous and having a finite number of maxima and minima on $a \leqq s \leqq b$, and a set of n axes formed in the manner already indicated. The graph of this function will cut the n axes in points by means of which we may define as above a linearly interpolated function $\bar{v}(s)$. If the given function $v(s)$ is held the same, and the system of axes is subjected to successive changes, so that the norm of the system decreases and approaches zero, it is clear that the resulting sequence of linearly interpolated functions $\bar{v}(s)$ approaches as a limit the function $v(s)$. The situation thus briefly described is typical of the character of the limiting process by which we shall repeatedly pass from an algebraic system to the corresponding transcendental equation or system.

The most interesting known oscillation and comparison theorems are those which arise in connection with linear homogeneous differential equations of the second order. Certain of the most fundamental properties of such an equation, namely, linearity, homogeneity, and that property in virtue of which the general solution may be expressed linearly in terms of two linearly independent particular solutions, are also fundamental properties of the algebraic system of n equations

$$\sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n), \tag{1}$$

in the $n+2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of the coefficients of this system being of rank n. Such a system possesses two linearly independent solutions u_i, v_i; and in terms of these the general solution may be written in the form $x_i = cu_i + dv_i$, where c and d are arbitrary constants. From the known theory of the differential equation, and by means of its relation to a particular form of system (1) as exhibited in the first section, we are led to certain properties of the solutions of this general system. We shall now state a few of these properties.

If we denote by Δ_{kl} the determinant of the matrix obtained from the matrix of coefficients in (1) by striking out the kth and lth columns, it may be shown without difficulty that Δ_{kl} and the determinant $u_kv_l - u_lv_k$ are both zero or neither zero, provided that when $\Delta_{kl} = 0$ we do not have the exceptional case in which $\Delta_{ml} = 0 = \Delta_{km}$ for every m of the set $1, 2, \cdots, n+2$ except k and l. A fairly straightforward argument, based on this elementary result, leads to a proof of the following fundamental theorem.

THEOREM. *Let $\Delta_{i,\,i+1}$ for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the treatment in the first paragraph of this section. Let u_i and v_i be two linearly independent solutions of the system* (1) *the matrix of whose coefficients is of rank n; and let these solutions be extended, by the method of linear interpolation employed above, to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other.*

If one examines the proof (see AMERICAN JOURNAL, vol. 43 (1921), p. 84) by which the foregoing theorem is established, he will see that it depends intimately upon the fact that the determinant $u_{i+1}v_i - u_iv_{i+1}$ is of one sign on R. If one undertakes to formulate a corresponding theorem for the more general system

$$\sum_{j=1}^{n+h} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n), \tag{2}$$

in the $n+h$ unknown quantities $x_1, x_2, \cdots, x_{n+h}$, where $h \geqq 2$,

the matrix of coefficients being again of rank n, he will find the situation in some respects the same as before, but in other (and perhaps more important) respects he will find it far different. If we let D_i denote the determinant of the nth order matrix obtained from the matrix of coefficients in (2) by striking out h consecutive columns beginning with the ith, and if we denote by w_i the determinant of order h,

$$w_i = \begin{vmatrix} x_i^{(1)} & x_{i+1}^{(1)} & \cdots & x_{i+h-1}^{(1)} \\ x_i^{(2)} & x_{i+1}^{(2)} & \cdots & x_{i+h-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_i^{(h)} & x_{i+1}^{(h)} & \cdots & x_{i+h-1}^{(h)} \end{vmatrix},$$

formed by means of a fundamental system of solutions $x_i^{(1)}$, $x_i^{(2)}$, $\cdots$, $x_i^{(h)}$ of (2), we shall find that for a range R of values of i on which D_i is of one sign it is true that w_i is of one sign; so that in this respect the situation is the same for all values of h.

But if we undertake to proceed further in the direction of a generalization of the theorem stated above, we find, not indeed that our steps are arrested, but that the theorem begins to lose its elegance and simplicity as soon as h is greater than 2, and that the complexity increases rapidly with increase of h. The marked simplicity for the case $h = 2$ is due in large measure to the fact that the expanded determinant w_i has but two terms when $h = 2$. It is possible to put in a variety of forms the complete generalization which emerges, but there seems to be no way in which we can proceed directly to the goal without a surrender of the elegance and simplicity of the theorem. The results which emerge are, however, not entirely without interest; in particular, they appear to point in the direction of theorems for differential equations (for instance) of general order h; but these will necessarily be rather complicated. We shall make no attempt to state any of the results here for general h, either for the case of the algebraic system or that of any of its limiting forms.

If the way of progress by such direct generalization is barred, we shall naturally seek the goal by some other means. Since the difficulty arises primarily from the fact that we have too many linearly independent solutions of (2), that is, too

many arbitrary constants in the general solution of (2), let us restrict attention to a particular class of solutions in which the number of arbitrary elements is 2. For the case of the algebraic system (2) the simplest way to do this is to adjoin $h - 2$ additional independent equations of the same type; and this brings us back to the case of system (1). But in the case of a differential or difference equation, for instance, such a means is not directly open to us. However, it is true that the boundary conditions in the transcendental cases are represented in the algebraic system by linear equations (see Bôcher's *Les Méthodes de Sturm*, loc. cit.). The suggestion then is to employ, for a difference or differential equation of order h, boundary conditions (usually $h - 2$ in number) so that there shall be but two arbitrary elements in the general solution subject to these boundary conditions. In § 5, we shall exhibit certain results suggested by these considerations.

In what follows in this section, we shall suppose that the notation in (1) is so chosen that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the lower right-hand corner of the matrix $||a_{ij}||$ are all different from zero. Without loss of generality, they may be taken to be positive, since, if they were not so, this could be brought about by changing the sign of every coefficient in certain of the equations in (1); and therefore we take them to be positive. We *assume* further that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the upper left-hand corner of the matrix $||a_{ij}||$ are all different from zero. Then it is possible to reduce system (1) to a new system

$$x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \quad (i = 1, 2, \cdots, n),$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}, and β_i is positive for all values $i = 1, 2, \cdots, n$. On writing $x_i = y_i u_i$, where y_1 and y_2 are positive quantities and

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3} \cdots \beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4} \cdots \beta_4\beta_2 y_2,$$

we obtain for the u_i the relations

$$(3) \qquad u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}},$$

where φ_i is a determinate function of the a_{ij} in (1). A second system of the form (1), under such hypotheses as we have just employed, would reduce to the normal form

$$v_{i+2} + \psi_i v_{i+1} + v_i = 0, \quad (i = 1, 2, \cdots, n). \tag{4}$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1). We state a few of the results for the normal forms (3) and (4).

THEOREM. *Let u_i and v_i be solutions of equations* (3) *and* (4), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coordinates. If $u(s)$ has consecutive zeros on the μth and $(m+1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(*a*) $\varphi_i \leqq \psi_i$, $(i = \mu, \mu+1, \cdots, m)$, *the equality sign not holding for all these values; or*

(*b*) $\varphi_i = \psi_i$, $(i = \mu, \mu+1, \cdots, m)$, *and the sets of constants u_i and v_i, for $i = \mu, \mu+1, \cdots, m$, are linearly independent.**

From this several further properties of comparison are readily derived. We state two of them.

Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$, $(i = 1, 2, \cdots, \nu)$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coordinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros (in increasing order) of $v(s)$ is to the left of the jth one of $u(s)$.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}},$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. Other similar theorems may be stated by modifying in certain respects the inequalities in the hypothesis and the conclusion.

* Compare the related theorem due to M. B. Porter, ANNALS OF MATHEMATICS, (2), vol. 3 (1902), p. 65.

3. *Algebraic Expansion Problems.* The r homogeneous linear algebraic systems in the unknown quantities x,

$$(5)\quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1 a_{1hij} + \lambda_2 a_{2hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$, and the r adjoint systems

$$(6)\quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1 a_{1hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

are consistent (in the sense that each system in each set of r systems has a solution not identically zero) for precisely the same values of the parameters $\lambda_1, \lambda_2, \cdots, \lambda_r$, namely, those values which satisfy the characteristic system of r determinantal equations

$$(7)\quad |a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}| = 0, \qquad (h = 1, 2, \cdots, r),$$

where for a given value of h the first member is the determinant of order n_h whose element in ith row and jth column is that which is written out explicitly.

The sets of characteristic values of $\lambda_1, \lambda_2, \cdots, \lambda_r$ for (5) and (6), namely, the sets of solutions of (7), necessarily finite in number if certain exceptional cases are avoided (as we intend they shall be), we shall denote by

$$\lambda_1^{(\rho)}, \lambda_2^{(\rho)}, \cdots, \lambda_r^{(\rho)}$$

for varying values of ρ, the two ordered sets being distinct for two distinct values of ρ. The corresponding solutions of (5) and (6) we shall then denote by

$$x_{hj}^{(\rho)}, y_{hj}^{(\rho)}, \qquad (j = 1, 2, \cdots, n_h;\ h = 1, 2, \cdots, r).$$

If we avoid certain exceptional cases, readily described in terms of the coefficients a in (5) and (6), we then have (for varying ρ and σ) the following relations expressing the fundamental conjugate character of the solutions of (5) and (6), namely:

$$(8)\quad \sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} \prod_{h=1}^{r} x_{i_h h}^{(\rho)} y_{h j_h}^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases}$$

where

$$D_{i_1j_1 \cdots i_r j_r} = \begin{vmatrix} a_{11j_1i_1} & a_{21j_1i_1} & \cdots & a_{r1j_1i_1} \\ a_{12j_2i_2} & a_{22j_2i_2} & \cdots & a_{r2j_2i_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1rj_ri_r} & a_{2rj_ri_r} & \cdots & a_{rrj_ri_r} \end{vmatrix}.$$

In what follows we shall assume that the mentioned exceptional cases are avoided, so that we shall have situations in which relations (8) are valid.

In order to bring out clearly the nature of these conditions of conjugacy, let us consider the case when $r = 1$. Systems (5) and (6) then take the form

$$(9) \quad \sum_{j=1}^{n} (c_{ij} + \lambda b_{ij})x_j = 0, \qquad \sum_{j=1}^{n} (c_{ji} + \lambda b_{ji})y_j = 0, \qquad (i = 1, 2, \cdots, n).$$

The conditions of conjugacy reduce to

$$(10) \quad \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}x_i^{(\rho)}y_j^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

In the more special case when $b_{ii} = b_i$, $b_{ij} = 0$ if $j \neq i$, these become

$$(11) \quad \sum_{i=1}^{n} b_i x_i^{(\rho)} y_i^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

When $b_i = 1$ for every i these become merely the usual conditions of biorthogonality; and these in turn reduce to the usual conditions of orthogonality in case we have $c_{ij} = c_{ji}$ and the same solutions x and y are taken for the two systems in (9). Thus we see that (8) affords an extensive generalization of a classic elementary relation of wide usefulness.

If m is the number of sets of characteristic values for (5) and (6), and if the set of constants $z_{i_1i_2 \cdots i_r}$ for i_h varying from 1 to n_h for each h, may be "expanded" in the form

$$(12) \quad z_{i_1i_2 \cdots i_r} = \sum_{k=1}^{m} c_k x_{1i_1}^{(k)} x_{2i_2}^{(k)} \cdots x_{ri_r}^{(k)},$$

where the c_k are independent of the subscripts i_h, the foregoing properties of conjugacy are available for an immediate determination of the c_k in the form

$$(13)\quad c_k = \frac{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} z_{i_1 i_2 \cdots i_r} \prod_{h=1}^{r} y_{h j_h}^{(k)}}{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} \prod_{h=1}^{r} x_{h i_h}^{(k)} y_{h j_h}^{(k)}}.$$

It is easy to obtain, in terms of the coefficients a of (5) and (6), broad sufficient conditions for the validity of the "expansion" in (12). The essential simplicity of these formulas is more readily apparent from the special case involved in equations (9) to (11). We assume that for this case m has its usual value n. Then if

$$(14)\qquad z_i = \sum_{k=1}^{n} c_k x_i^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

we have for c_k one or the other of the values

$$(15)\qquad \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} z_i y_j^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} x_i^{(k)} y_j^{(k)}}, \qquad \frac{\sum_{i=1}^{n} b_{ii} z_i y_i^{(k)}}{\sum_{i=1}^{n} b_{ii} x_i^{(k)} y_i^{(k)}},$$

according as we have the case of (10) or (11). If $b_i = 1$ the latter reaches the maximum of elegance, at least if equations (9) are then self-adjoint and the same solutions x and y of the two systems are taken.

In the foregoing expansion formulas the number of subscripts on the "function" to be expanded in terms of the "functions," x is equal to the number of parameters λ involved in the original system of algebraic equations. When we proceed to the transcendental limiting cases we shall see that these algebraic results are in the form best suited to applications to equations involving functions of one variable. In order to obtain a form suitable as the heuristic guide in problems involving functions of more than one variable in the original equations, it is desirable to look upon our expansion formulas in a way slightly different from that in evidence in the foregoing work. We can best bring out what is needful by considering expansion (14). Let us suppose that n is the product $\mu\nu$ of two integers. Let us replace the subscript i, running over the set $1, 2, \cdots, n$, by the double subscript ij where i

runs over the set 1, 2, $\cdots$, μ and j over the set 1, 2, $\cdots$, ν. Then equations (14) become

$$(16)\quad z_{ij} = \sum_{k=1}^{\mu\nu} c_k x_{ij}^{(k)}, \qquad (i = 1, 2, \cdots, \mu;\ j = 1, 2, \cdots, \nu).$$

The coefficients c_k are of course determined by the same methods as before but the formulas are modified through the replacing of single summations by double summations. It is clear that we may pass in a similar way to multipartite instead of bipartite subscripts. This obvious remark concerning a change from one subscript to more than one subscript entails important consequences in a great variety of expansion problems.

With a given set of n axes on an interval $a \leqq s \leqq b$, as in the opening paragraphs of § 2, let us extend by linear interpolation the n sets $x_i^{(k)}$, $k = 1, 2, \cdots, n$, and the set z_i of equation (14); and let $x^{(k)}(s)$, $k = 1, 2, \cdots, n$, and $z(s)$ denote the functions of the continuous variable s so obtained. Then it is easy to prove that

$$(17)\qquad z(s) = \sum_{k=1}^{n} c_k x^{(k)}(s),$$

where the coefficients c_k have the same values as in (14). Thus we pass from "expansions" of sets of constants to interesting expansions of a particular class of functions of a continuous variable.

Let us consider the like matter for functions of two subscripts and expansions of the form (16). For the representation of u_{ij} for $i = 1, 2, \cdots, \mu$ and $j = 1, 2, \cdots, \nu$, we shall start from ν parallel planes, one for each value of j. In each of these we use for s the same range $a \leqq s \leqq b$ and place the planes in order 1, 2, $\cdots$, ν for j, each directly in front of the preceding one and arrange the vertical axes, so that the vertical axes in any one plane are (point for point) the orthogonal projections on that plane of the axes in any other plane. Next, for fixed j we extend the u_{ij} to $u_j(s)$ by linear interpolation as before. Then for each value of s we connect the points $u_j(s)$, $j = 1, 2, \cdots, \nu$, by straight-line segments joining the consecutive points. We thus get a sort of broken surface affording

the graphical representation of a function $u(s, t)$, gotten (we may say) by linear interpolation from the points u_{ij}. We call this the graphical representation of the set of constants u_{ij} with respect to the given system of axes. Let $c \leqq t \leqq d$ denote the range of t in this representation, this range evidently depending on the positions of the ν planes employed in setting up the graphical representation.

Let us replace the subscripts i, j in the functions in (16) by the continuous variables s, t in accordance with the method just indicated. Then it is easy to prove that we have

$$(18) \qquad z(s, t) = \sum_{k=1}^{\mu\nu} c_k x^{(k)}(s, t), \qquad (a \leqq s \leqq b, c \leqq t \leqq d),$$

where the coefficients c_k have the same values as in (16). Similar (but more complicated) extensions may be associated with (12) so that we come through to a relation of the form

$$(19) \quad z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{m} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r),$$

where the coefficients c_k have the same values as in (12) and the functions involved depend on continuous variables. Further generalizations may also be made in (12) and in the last formula by replacing one or more of the subscripts i_h in (12) by multipartite subscripts and by linear interpolation in the resulting relations.

In the case of the principal transcendental expansion problems the foregoing expansions in a finite number of terms are replaced by expansions in an infinite number of terms so that difficult questions of convergence arise. In one of the most important of these transcendental problems, namely, that treated by Birkhoff (TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), pp. 373–395), the convergence questions are dealt with by the aid of a contour integral by means of which the sum of any finite number of terms of the series is readily expressed. Similar contour integrals exist for representing the sum of a finite number of terms of the series in (12) or (14). Let us consider the general case. For any given value of h let $\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the determinant in the first member of (7) and let $\Delta_{hij}(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the cofactor of the element in the

ith row and jth column of this determinant. We assume for the present purpose that systems (5) and (6) are so restricted that the solutions of each for characteristic values of the λ's is unique (except for a constant factor) and that for each h

$$\lim_{\lambda_1=\lambda_1^{(k)},\cdots,\lambda_r=\lambda_r^{(k)}} \frac{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\lambda_h - \lambda_h^{(k)}}$$

exists and is finite, where $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$ is any set of characteristic values for (5) and (6). Then, if we avoid a certain exceptional condition not arising in the (most important) case $r = 1$, we have for the kth term of the expansion of $z_{t_1 t_2 \cdots t_r}$ afforded by (12) the value

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^r \int_{\Gamma_{1k}} \int_{\Gamma_{2k}} \cdots \int_{\Gamma_{rk}} \left\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \right.$$

$$\left. \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1 j_1} \cdots {}_{i_r j_r} z_{i_1 i_2 \cdots i_r} G_{j_1 t_1 \cdots j_r t_r} \right\} d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where Γ_{hk} $(h = 1, 2, \cdots, r)$ is a contour in the λ_h-plane about the point $\lambda_h^{(k)}$, containing in its interior no other characteristic value of λ_h, and where

$$G_{j_1 t_1 \cdots j_r t_r} = \prod_{h=1}^{r} \frac{\Delta_{h j_h t_h}(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}.$$

If we replace the contours Γ_{hk} $(h = 1, 2, \cdots, r)$ by Γ_h, a contour which includes within it all the characteristic values of λ_h, and perform the same multiple integration about such contours, we shall have the value of the function $z_{t_1 t_2 \cdots t_r}$. It is clear that we may form similarly the contour integral for any given partial sum of the series for $z_{t_1 t_2 \cdots t_r}$ in (12). For the case of systems (9) we have the equations

$$z_t = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Delta_{jt}(\lambda)}{\Delta(\lambda)} b_{jt} \hat{z}_i d\lambda, \qquad (t = 1, 2, \cdots, n),$$

where Γ is a contour in the λ-plane inclosing all the characteristic values for (9), and where $\Delta(\lambda)$ is the determinant of the coefficients in the first member of (9) and $\Delta_{jt}(\lambda)$ is the cofactor of the element in the jth row and tth column of $\Delta(\lambda)$.

For use in connection with expansion problems involving difference equations, it is desirable to observe that several of the important formal properties of (5) and (6) are preserved in the case of infinite systems in which appropriate conditions

of convergence are realized. Thus in (5) and (6) we may allow some or all of the numbers $n_1, \cdots, n_r$ to become infinite; for the sake of simplicity we suppose that all of them become infinite. The system (7) is replaced by a transcendental system having, in the cases which interest us, an infinite number of solutions. Subject to suitable limitations for procuring convergence, we may proceed as before and thus derive a new form of (8) in which the change is that of replacing each of the numbers $n_1, \cdots, n_r$ by ∞, and a new form of (12) in which m is replaced by ∞ and the relations are to be valid for each i_h ranging over the set 1, 2, 3, $\cdots$, the expressions for the coefficients c_k $(k = 1, 2, 3, \cdots)$ having the form which results from that in (13) on replacing $n_1, \cdots, n_r$ each by ∞. Analogues of many of the additional foregoing results persist. In the formulas thus obtained we have a useful heuristic guide to certain expansion problems in the theory of difference equations, a part of which we shall presently indicate.

4. *Transcendental Expansion Problems.* Let us consider a fixed interval $a \leqq s \leqq b$ of the real s-axis, and let us associate with certain points of this interval the discrete values of i and j in equations (5) and (6). For the hth system in either (5) or (6), we take on (ab) a set of points n_h in number including the points a and b; and we interpolate x_{hj} into a function $x_h(s)$ by the method of linear interpolation described above. Similarly a_{khij} is interpolated into a function $a_{kh}(s, t)$. Equations (5) and (6) may now be looked upon as establishing relations among the functional values of these functions of s and t at certain points only of the axes of s and t; and so of defining the solution functions at these points and these alone, their definitions being completed by the methods of interpolation agreed upon.

To these systems we now apply certain limiting processes, allowing the numbers n_h (or at least a part of them) to increase indefinitely. As they increase, the functions $x_h(s)$ and $a_{kh}(s, t)$ pass through a corresponding sequence of changes. If the processes involved lead to replacing the original equations by

well-defined limiting equations and their solutions by well-defined functions, we have in the process a suggestion of a heuristic guide to probable solutions of the limiting problems and to certain probable fundamental properties of these, together with intimations as to how they shall be established.

Usually we shall require that the distribution of basic points on the interval (ab) of the s-axis shall undergo change in such way that the norm of the distribution shall approach zero. By such processes one may realize, for instance, integral equations and differential equations as limiting cases. Among the characteristic results, for the finite case, which persist after certain limiting operations of this type have been performed are those relating to conjugacy, expansions, and representation of the latter by contour integrals. Without going into details, we may state the general limiting forms of certain equations, special cases of which we shall have occasion to consider. Equations (8), (12), (13) pass into the following limiting forms:

$$\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r)$$

$$\prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\sigma)}(t_h) ds_1 dt_1 \cdots ds_r dt_r \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{20}$$

$$z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{\infty} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r), \tag{21}$$

$$c_k = \frac{\int_a^b \int_a^b \cdots \int_a^b z(s_1, s_2, \cdots, s_r) D(s_1, t_1, \cdots, s_r, t_r) y_1^{(k)}(t_1) y_2^{(k)}(t_2) \cdots y_r^{(k)}(t_r) ds_1 dt_1 \cdots ds_r dt_r}{\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \prod_{h=1}^{r} x_h^{(k)}(s_h) y_h^{(k)}(t_h) ds_1 dt_1 \cdots ds_r dt_r}. \tag{22}$$

Corresponding to (10), (11), (14), (15) we have special cases of importance which indicate more clearly the essential simplicity of the formulas, namely,

$$\int_a^b \int_a^b b(t, s) x^{(\rho)}(s) y^{(\sigma)}(t) ds dt \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{23}$$

$$\int_a^b b(s) x^{(\rho)}(s) y^{(\sigma)}(s) ds \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{24}$$

$$z(s) = \sum_{k=1}^{\infty} c_k x^{(k)}(s), \qquad (a \leqq s \leqq b), \tag{25}$$

$$c_k = \frac{\int_a^b \int_a^b b(t, s) z(s) y^{(k)}(t) ds dt}{\int_a^b \int_a^b b(t, s) x^{(k)}(s) y^{(k)}(t) ds dt} \tag{26}$$

or

$$\frac{\int_a^b b(s) z(s) y^{(k)}(s) ds}{\int_a^b b(s) x^{(k)}(s) y^{(k)}(s) ds}.$$

If for the last form of c_k we have $c(s, t) = c(t, s)$, so that $y^{(k)}(s)$ may be taken equal to $x^{(k)}(s)$, and if $b(s) \equiv 1$, we have in (25) and the latter form of c_k in (26) formulas for the formal expansion of an "arbitrary" function of a single variable in terms of orthogonal functions of that variable; so that all the formulas (20) to (26) are generalizations of classic relations in the expansion of functions.

In these formulas the number of variables in the function z to be expanded is equal to the number of parameters involved in the problem. This correspondence is not essential. The desired extension can best be brought out by starting from the particular expansion (16). By linear interpolation we first obtain (18). Then we may proceed to the limiting case in such wise that μ and ν simultaneously approach infinity, the norm of the corresponding distributions of points on the s-axis and the t-axis approaching zero. We are thus led heuristically to an expansion of the form

$$z(s, t) = \sum_{k=1}^{\infty} c_k x^{(k)}(s, t), \tag{27}$$

where the x's are now solutions of the limiting problem. It is easy to see that the properties of conjugacy are maintained formally and that we may therefore readily determine the coefficients c_k. We may also proceed from (18) to the limit in another way, namely, by holding μ fixed and allowing ν to become infinite as before; we are thus led to expansions of the form

$$z_i(s) = \sum_{k=1}^{\infty} c_k x_i^{(k)}(s), \quad (i = 1, 2, \cdots, \mu), \tag{28}$$

for expanding a system of μ given functions in terms of μ sets

of functions, the coefficients c_k of the expansion being the same for each of the given functions. This remarkable type of expansion I saw first in the manuscript of Dr. C. C. Camp's dissertation which he was kind enough to allow me to read; it occurred there in connection with a particular system of two linear differential equations of the first order.* In (27) we have an expansion of the type which arises in the theory of partial differential equations in two independent variables. In (28) we have the type of expansions which arises in the theory of n linear differential equations of the first order. It is clear that the type of extension employed in this paragraph for the case of one parameter may be utilized in a variety of ways in connection with problems involving r parameters.

Let us consider the results to which this heuristic guide leads us in the case of the adjoint differential systems

$$(29) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij})y_j, \qquad (i = 1, 2, \cdots, n),$$

$$(30) \qquad \frac{dz_i}{dx} = \sum_{j=1}^{n} (-a_{ji} - \lambda\alpha_{ji})z_j, \qquad (i = 1, 2, \cdots, n).$$

If for fixed i we multiply these respectively by z_i and y_i, add the resulting equations member by member, sum as to i from 1 to n, and then integrate from a to b (a range in which the coefficients are assumed to be continuous) we have

$$(31) \qquad [y_1z_1 + y_2z_2 + \cdots + y_nz_n]_{x=a}^{x=b} = 0.$$

The first member of this relation is a non-singular bilinear form in the two sets of $2n$ variables each,

$$(32) \qquad \begin{matrix} y_1(a), y_2(a), \cdots, y_n(a), y_1(b), \cdots, y_n(b); \\ z_1(a), \cdots, z_n(a), z_1(b), \cdots, z_n(b). \end{matrix}$$

It can be written in an infinite number of ways in the form

$$(33) \qquad [y_1z_1 + \cdots + y_nz_n]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} Y_i(y)Z_i(z),$$

where $Y_i(y)[Z_i(z)]$ is a set of $2n$ linearly independent homo-

* Since this was written a paper by A. Schur has appeared dealing with this problem; see MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 213–236. It contains (p. 214) a reference to a special case of the problem treated by Hilbert (GÖTTINGER NACHRICHTEN, 1906, pp. 474–480). This I had not seen before.

geneous linear functions of the $2n$ variables $y[z]$ of (32). With (29) and (30), respectively, we associate the boundary conditions

$$Y_i(y) = 0, \qquad (i = 1, 2, \cdots, n), \tag{34}$$

$$Z_i(z) = 0, \qquad (i = n+1, \cdots, 2n). \tag{35}$$

Then the characteristic values λ for problem (29), (34) are the same as those for problem (30), (35). We suppose that the conditions are set up in such a way that the number of these characteristic values is infinite. We denote the characteristic values and the corresponding solutions by $\lambda^{(k)}$, $y_i^{(k)}$, $z_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. Corresponding to (20), (21), (22) we now have the relations

$$\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} y_i^{(k)} z_j^{(l)} \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^{\infty} c_k y_i^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} f_i z_j^{(k)} dx}{\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} y_i^{(k)} z_j^{(k)} dx}, \qquad (k = 1, 2, 3, \cdots).$$

If we apply to systems (29) and (30) a limiting process often employed by Volterra we are led to adjoint integro-differential equations of the form

$$\frac{\partial y(x, s)}{\partial x} = \int_\alpha^\beta \{u(x, s, t) + \lambda v(x, s, t)\} y(x, t) dt,$$

$$\frac{\partial z(x, s)}{\partial x} = \int_\alpha^\beta \{- u(x, t, s) - \lambda v(x, t, s)\} z(x, t) dt,$$

where the range of variation of x is from a to b while that of s and t is from α to β. Corresponding to the last three equations of the preceding paragraph we now have the following:

$$\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_l(x, t)\, ds\, d\, t dx \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f(x, s) = \sum_{k=1}^{\infty} c_k y_k(x, s),$$

$$c_k = \frac{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) f(x, s)\, z_k(x, t)\, ds\, dt\, dx}{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_k(x, t)\, ds\, dt\, dx}, \qquad (k = 1, 2, \cdots),$$

where $y_k(x, s)$, $z_k(x, s)$ denote the solutions corresponding to the characteristic value λ_k of the set $\lambda_1, \lambda_2, \cdots$ of distinct characteristic values.

The same procedure may be applied with equal facility to the adjoint systems of integro-differential equations

$$\frac{\partial y_i(x, s)}{\partial x} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij})\, y_j(x, s) + \int_\alpha^\beta \sum_{j=1}^{n} [\rho_{ij}(x, s, t) + \lambda\sigma_{ij}(x, s, t)]\, y_j(x, t)\, dt,$$

$$\frac{\partial z_i(x, s)}{\partial x} = \sum_{j=1}^{n} (-a_{ji} - \lambda\alpha_{ji})\, z_j(x, s) + \int_\alpha^\beta \sum_{j=1}^{n} [-\rho_{ji}(x, t, s) - \lambda\sigma_{ji}(x, t, s)]\, z_j(x, t)\, dt,$$

for $i = 1, 2, \cdots, n$, and to various generalizations and extensions of them. In this way emerge formal properties of various types of expansions arising in connection with differential and integro-differential equations. The problems may likewise be set up with equal facility for the case of an equation or system of equations of any order with respect to differentiation instead of merely for the first order as in the foregoing problems. Furthermore, one can treat equally well a system with r parameters λ involved in a way analogous to that observed in connection with the algebraic problem defined at the beginning of § 3. The formulas necessarily become more complicated, but the fundamental guiding ideas are unmodified; the algebraic theory indicates the whole procedure and suggests the principal results as limiting cases of the algebraic propositions.

The way in which the corresponding problem may be set up for the difference equation will be indicated by a very brief statement. Let us consider the adjoint systems of difference equations

$$(36)\quad u_i(x + 1) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})\, u_j(x), \qquad (i = 1, 2, \cdots, n),$$

$$(37)\quad v_i(x) - v_i(x + 1) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})\, v_j(x + 1), \qquad (i = 1, 2, \cdots, n),$$

where φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and vanish there to an order at least as high as the second. We confine attention to those solutions alone which have the property that each function in a solution approaches a (finite) limiting value as x approaches infinity along any ray from the origin exclusive of the negative axis of imaginaries or along any line proceeding to the right parallel to the axis of reals.

If we multiply (36) through by $v_i(x+1)$ and (37) through by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \Delta\{u_i(x)v_i(x)\} = 0. \tag{38}$$

If the real part of a is sufficiently large we have u_i and v_i analytic at every point x whose real part is not less than the real part of a. Hence in (38) we may sum as to x from a to infinity, where x runs over the values $a, a+1, a+2, \cdots$; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)\, v_i(\infty) - u_i(a)\, v_i(a)\} = 0. \tag{39}$$

Let us now suppose that adjoint homogeneous linear boundary conditions, implying (39), are set up on the $u_i(\infty)$, $u_i(a)$ and on the $v_i(\infty)$, $v_i(a)$ similar to conditions (34) and (35) in a similar problem above and let us suppose that we have the infinite set of characteristic values and corresponding solutions $\lambda^{(k)}$, $u_i^{(k)}$, $v_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. The fundamental formulas for the expansion problem thus arising are the following:

$$\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(l)}(a+1+t) \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, f_i(a+t)\, v_j^{(k)}(a+1+t)}{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(k)}(a+1+t)}, \qquad (k = 1, 2, 3, \cdots).$$

If a classic limiting process of Volterra is applied to systems (36) and (37), they go over into adjoint integro-difference equations analogous to the integro-differential equations treated above; and these may be generalized to related systems just as we generalized the corresponding problem for integro-differential equations. In all these cases the properties of conjugacy and the formal results for expansions persist in the form naturally to be expected. Moreover, corresponding and closely similar theories exist for q-difference and integro-q-difference equations.

In view of the basic algebraic theory and the transcendental problems already treated it is clear that there must exist in the theory of integral equations expansion problems involving r parameters not only in the classic case when $r = 1$ but also in the general case when r is any positive integer. Moreover, if we think of the several types of expansion problems—those for differential, difference, q-difference, integral, integro-differential, integro-difference, and integro-q-difference equations—in intimate connection with the basic algebraic theory, it becomes apparent that the case of r parameters (for $r > 1$) is not confined to a set of r equations of the same sort. There is nothing to prevent one subset of the basic algebraic equations from proceeding to differential equations as limiting forms, another to difference equations, another to integral equations, another to q-difference equations, and so on. Thus we can see beforehand that we may formulate the expansion problem for a variety of mixed systems. One is in fact led naturally to such systems in the consideration of certain integro-differential, integro-difference, and integro-q-difference equations. We shall not take space to treat any of these mixed systems, preferring rather to exhibit briefly the nature of the problem for partial differential equations. From these one may proceed naturally to related integro-differential equations. Similar problems may be formulated for partial difference and integro-difference equations.

Let us consider the adjoint partial differential equations

$$L(u) + \lambda L_1(u) = 0, \qquad M(v) + \lambda M_1(v) = 0,$$

where

$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u,$$

$$L_1(u) \equiv p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru,$$

the symbols u, v, a, b, c, α, β, γ, p, q, r denoting functions of the independent variables x and y continuous and suitably differentiable in the square $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$. It is easy to show the existence of an identity of the form

$$\int_0^1\int_0^1 [v\{L(u)+\lambda L_1(u)\} - u\{M(v)+\lambda M_1(v)\}]dxdy \equiv \int_0^1 B dt,$$

where B is a sort of bilinear form in the functions u and v and their first derivatives, the arguments of the functions being suitably restricted. If suitable boundary conditions are set up for the u-problem and for the v-problem so that $B \equiv 0$ in virtue of the boundary conditions and so that we have the infinite set of characteristic values and corresponding solutions λ_k, u_k, v_k, $(k = 1, 2, 3, \cdots)$, then we have the following fundamental formulas:

$$\int_0^1\int_0^1 u_i M_1(v_j)\, dx\, dy \begin{cases} = 0 \text{ if } i \neq j, \\ \neq 0 \text{ if } i = j, \end{cases}$$

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y),$$

$$c_k = \frac{\int_0^1\int_0^1 f(x, y)\, M_1(v_k)\, dx\, dy}{\int_0^1\int_0^1 u_k(x, y)\, M_1(v_k)\, dx\, dy}, \quad (k = 1, 2, 3, \cdots).$$

These results are readily carried over to partial differential equations of other forms and to the case of much more general regions than the square over which we have integrated in this particular instance.

5. *Transcendental Oscillation and Comparison Theorems.* The fundamental algebraic oscillation theorem in the earlier part of § 2 has several limiting forms of interest. We consider first those for homogeneous linear differential equations. The result is classic for equations of the second order: the zeros of two linearly independent solutions of such an equation separate each other throughout any interval containing no singular point of the equation. We shall now state one

extension of this result to equations of order n, $n > 2$. Such an equation we write in the general form

$$u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1} u' + p_n u = 0, \tag{40}$$

where the superscripts refer to differentiation and where the coefficients p are real-valued, single-valued, continuous functions of the real variable x on the interval $(a\ b)$. Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude linearly dependent on two linearly independent solutions (this being done so that the new theorem shall indeed be a direct limiting form of the algebraic theorem referred to). Suitable boundary conditions may be expressed by means of Stieltjes integrals in the form

$$\sum_{j=1}^{\nu} \int_a^b L_{ij}(u)\, d\psi_{ij}(x) = 0, \tag{41}$$

$$(i = 1, 2, \cdots, n-2;\ \nu = \text{positive integer}),$$

where the $\psi_{ij}(x)$ are functions of bounded variation on (ab) and the $L_{ij}(u)$ denote homogeneous linear expressions in u, u', $\cdots$, $u^{(n-1)}$. As a special case we have conditions which reduce to the following: $u(a) = 0, u'(a) = 0, \cdots, u^{(n-3)}(a) = 0$.

By aid of a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (40) we define constants λ through the formulas

$$\sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j)\, d\psi_{ik}(x) = \lambda_{ij}, \qquad \begin{pmatrix} i = 1, 2, \cdots, n-2 \\ j = 1, 2, \cdots, n \end{pmatrix},$$

and then the determinant $D(x)$,

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_{n-2,1} & \lambda_{n-2,2} & \cdots & \lambda_{n-2,n} \end{vmatrix}.$$

The zeros of $D(x)$ are independent of the choice of the fundamental system by means of which they are defined; that is, they depend only on (40) and (41). These zeros we shall call the *special points* of (ab) for the problem (40), (41). Then we have the following theorem:

On any interval of (ab) containing no special points for the

problem (40), (41) *the zeros of any two linearly independent solutions of* (40), (41) *separate each other.*

The foregoing result may be applied to the case of any two linearly independent solutions of (40) without reference to any *preassigned* boundary conditions. For this purpose we associate with any two linearly independent solutions u_1, u_2 of (40) a set of boundary conditions capable of representation in the form (41) and having the property that the solutions of (40) and the determined conditions (41) are those functions and those alone which are linearly dependent upon u_1 and u_2. As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad (k = 0, 1, 2, \cdots, n-1;\ i = 1, 2).$$

Let the coefficients σ_{ij} be so chosen that the equations

$$\sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \sigma_{i2}u''(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad (i = 1, 2, \cdots, n-2),$$

have those solutions and those alone which may be written in the form $u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}$, $(k = 0, 1, \cdots, n-1)$, where a_1 and a_2 are arbitrary constants. Having thus introduced suitable boundary conditions restricted by means of the given solutions u_1 and u_2, we may define the determinant $D(x)$, and hence the special points, in the way indicated for the preceding case. By means of these points we may divide the interval (ab) into segments on the interior of each of which the zeros of u_1 and u_2 separate each other in accordance with the foregoing theorem.

As a second case, let us consider the difference equation

$$L(x)\,u(x) + M(x)\,u(x+1) + N(x)\,u(x+2) = 0$$

in which all the indicated functions are real-valued, single-valued, continuous functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation and let a $(a \geqq \alpha)$ be a point for which $w(a) \neq 0$ where $w(x) = u_1(x)\,u_2(x+1) - u_1(x+1)\,u_2(x)$. Let $\bar{u}_i(x)$ be the function obtained by linear interpolation from the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$,

with respect to a system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points a, $a+1$, $a+2$, $\cdots$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the characteristic points of $u_i(x)$ with respect to a. Then we have the following theorem analogous to that of Sturm for a second order differential equation: *The characteristic points of $u_1(x)$ and $u_2(x)$ with respect to a separate each other.*

This result admits of extension to a system formed of a difference equation of order n, $n > 2$, and $n-2$ boundary conditions of a certain general sort restricting the simultaneous solutions to a two-fold infinitude. Similar results may also be obtained for q-difference equations. In fact, those for difference and q-difference equations are both special cases of like results for a rather general class of functional equations including difference and q-difference equations as special cases; but we shall not here take the space necessary to set forth these more general results. The comparison theorems which follow suggest their nature. The second theorem of § 2 has as a limiting case a theorem which is essentially equivalent to the following classic Sturmian theorem of comparison.

Let us consider the two differential equations

$$\frac{d}{dx}(K_1u') - G_1u = 0, \tag{42}$$

$$\frac{d}{dx}(K_2u') - G_2(u) = 0, \tag{43}$$

in which K_1, K_2, G_1, G_2 are functions which are continuous throughout the interval (ab) defined by the inequalities $a \leqq x \leqq b$ and in this interval satisfy the relations

$$0 < K_2 \leqq K_1, \qquad G_2 \leqq G_1. \tag{44}$$

Moreover, let u_1 and u_2 be solutions of (42) and (43), respectively, neither of which is identically zero in (ab). Then if x_1 and x_2 are any two consecutive zeros of u_1 in (ab) there is at least one zero of u_2 in the interior of the interval (x_1x_2) provided either that at least one equality sign in (44) fails to hold at every point of the interval (x_1x_2), or that u_1 and u_2 are linearly independent in (x_1x_2).

The two results stated in the paragraphs following the theorem referred to in § 2 have as limiting cases two theorems which are also classic in the theory of Sturm. The algebraic theorems were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k (ANNALS OF MATHEMATICS (2), vol. 19 (1918), pp. 159–171). It is possible to extend the general algebraic results of the latter part of § 2 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving some of these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the substitution $x' = S_x$, denoting its nth power by $x' = S_x^n$. Let it be such that there exists an open interval I of the real x-axis, such that $\lim S_{x_0}^n = \beta$ for every x_0 of I, β being an end-point of I and the limit being approached monotonically. Then consider the functional equations

$$u(S_x^2) + \varphi(x)u(S_x) + u(x) = 0, \tag{45}$$

$$v(S_x^2) + \psi(x)v(S_x) + v(x) = 0, \tag{46}$$

in which $\varphi(x)$ and $\psi(x)$ are real-valued, single-valued, continuous functions of the real variable x on the interior of the interval I. Suppose, furthermore, that S_x is such that each of these equations has a fundamental system of solutions consisting of two functions which are real-valued, single-valued, and continuous on the open interval I. (In case $S_x \equiv x + 1$ and I is the interval $\alpha < x < \infty$, our equations are ordinary difference equations; in case $S_x = qx$, q being real and greater than unity, and I is the interval $0 \leqq \alpha < x < \infty$, our equations are q-difference-equations.) If a is an interior point of the interval I, we define the characteristic points of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $\bar{t}(x)$ derived from

the constants $t(a)$, $t(S_a)$, $t(S_a^2)$, $\cdots$ by linear interpolation with respect to the system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points $a, S_a, S_a^2, \cdots$. Let u and v be real-valued, single-valued, continuous solutions of equations (45) and (46), respectively. Then we have the following three theorems:

If $u(x)$ has two consecutive characteristic points with respect to a on the μth and $(m+1)$th intervals $(\mu < m)$ of the set of intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, S_a^3, \cdots$, then $v(x)$ has a characteristic point between these characteristic points of $u(x)$ provided that either

(a) $\varphi(x) \leqq \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, the equality sign not holding for all the end-points of these intervals; or,

(b) $\varphi(x) = \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, and the two sets of constants

$$u(S_a^{\mu-1}),\ u(S_a^{\mu}),\ \cdots,\ u(S_a^{m-1}),$$
$$v(S_a^{\mu-1}),\ v(S_a^{\mu}),\ \cdots,\ v(S_a^{m-1}),$$

are linearly independent.

Next, let us suppose that $u(a) \neq 0$, $v(a) \neq 0$, $u(S_a)/u(a) > v(S_a)/v(a)$; and that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, S_a^2, \cdots, S_a^{\nu-1}$. If $u(x)$ has k characteristic points on the ν intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, \cdots, S_a^{\nu}$, then $v(x)$ has at least k characteristic points on these intervals; and the jth of these characteristic points of $v(x)$ (counted from a towards S_a^{ν}) is nearer to a than the jth characteristic point of $u(x)$.

In the third place, let $u(a), v(a), u(S_a^k), v(S_a^k)$ be all different from zero and let $u(S_a)/u(a) > v(S_a)/v(a)$. Let $u(x)$ and $v(x)$ have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, \cdots, S_a^k$. Then we have

$$\frac{u(S_a^{k+1})}{u(S_a^k)} > \frac{v(S_a^{k+1})}{v(S_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, \cdots, S_a^{k-1}$.

THE UNIVERSITY OF ILLINOIS

HUYGENS' DIOPTRICA

Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences. Tome treizième. *Dioptrique.* 1653; 1666; 1685–1692. La Haye, Martinus Nijhoff, 1916. pp. i–clxviii; 1–905. 4to.

The monumental edition of the correspondence and writings of Christiaan Huygens was undertaken in 1888 by the Dutch Society of Sciences. The volume here under review, containing Huygens's original contributions to dioptrics and comprising with introduction, notes, etc., more than a thousand pages, is in every way worthy of its illustrious author. Unstinted praise should be bestowed on both the editor and the publisher, who have spared no pains in making this invaluable work as authoritative, complete and convenient as possible.

The manuscript of Huygens's writings on *Dioptrica* consists of detached sheets, 166 in all, the oldest portions of which date back to 1652, when Huygens was 23 years old. Almost until the end of his life in 1695, he was continually adding to it, revising it and sometimes planning to rewrite it entirely. By the end of 1653 he had composed a first draft of his *Tractatus de refractione et telescopiis* (pp. 1–271). At intervals during the following twelve years he was often at work on the same material, but for one reason or another its publication was constantly delayed. About 1665 he began to study more systematically the theory of spherical aberration; the results of these investigations are contained in the second part of the *Dioptrica* under the title of *De aberratione radiorum a foco* (pp. 273–353). About 1672, or perhaps a little later, we hear of his being again at work on his treatise on dioptrics with renewed ardour. By this time Newton's explanation of the phenomena of dispersion had been published. Its importance and its bearing on his own problems was quickly perceived by Huygens to the extent of modifying some of his views. Meanwhile, the undulatory theory had been born in his mind, and he was planning a more extensive treatise involving a revision and rearrangement of his previous manuscripts on dioptrics. In 1677 he found the explanation of double refraction in Iceland spar, which he naturally regarded as the most beautiful confirmation of his new theory of light. In comparison his earlier work in dioptrics seemed to him of secondary importance. Consequently, he determined to publish first a treatise on the wave theory of light with its principal applications, but without entering in detail into the theory of mirrors and lenses. This was the origin of his famous *Traité de la lumière.* Although it was not published until 1690, it was practically completed in 1678, and had been read before the Academy of Sciences in Paris in 1679. The third part of the *Dioptrica*, entitled *De telescopiis et microscopiis* (pp. 443–511), seems to have been composed in 1685. Apparently Huygens could never make up his mind to publish his optical theories and researches because he was continually adding fresh discoveries and gaining new insight and new points of view. In 1692 the work was still unfinished and at that time Huygens writes to Leibnitz: "il y a bien des choses

à demesler dans cette Dioptrique, et il s'en est offert tousjours de nouvelles, jusqu'à cette heure, qu'il me semble d'avoir tout penetré, quoy que je n'aye pas encor achevé de tout escrire." And so, as a matter of fact, Huygens's *Dioptrica* was never actually completed, and was not published at all until after his death. The greater part of it was included in De Volder and Fullenius's edition of Huygens's *Opuscula posthuma* published in Leiden in 1703. Unfortunately, not a few of Huygens's most valuable theorems never saw the light until long after they were obtained. Consequently he lost the priority of a number of important discoveries.

In this volume the text of the *Dioptrica* is given in Latin with facsimile cuts of Huygens's original diagrams and illustrations. On the opposite page there is a French translation. In an introduction of 167 pages, the editor has reviewed the entire work in detail. The text itself is clarified and expounded by notes and comments. In short the reader has every aid that could be desired, including a complete and accurate index of the whole contents.

One of the chief fascinations in a lifelong work like the *Dioptrica* is to trace in it the voyages of a great and original mind. The reader is continually surprised not only by the impresses of genius which have been left on these pages, but perhaps most of all by the almost startling modernness of many of Huygens's conceptions, as was long ago remarked by the late Professor Silvanus P. Thompson, reminding us again and again how little in advance of men like Newton and Huygens we are even in the twentieth century. The outburst of optical science which followed the invention of the telescope, and which constitutes a kind of Elizabethan Era in the history of scientific discovery, seems, so to speak, to flash forth anew from between the covers of this volume, and we catch some spark of the joyous enthusiasm that possessed those eager men who first employed the telescope and the microscope. Somehow too from an occasional scornful allusion to the opinions of some of his predecessors or contemporaries, at least one reader has been led to suspect that Huygens had a human side also and did not suffer fools gladly!

Amid such a mass of material, it is difficult to single out one thing for special comment rather than another. Certainly it is worth noting with what elegance and skill—far in advance of his contemporaries in this respect—Huygens derives from the law of refraction the fundamental characteristics of optical imagery in the limiting case when the effective rays are nearly normal to the refracting surface. If these propositions had been published about 1653 when they were first obtained by Huygens, he would certainly have had the priority for them. They were communicated in an anagram to the Royal Society in 1669, but at that very time Dr. Isaac Barrow's *Lectiones opticæ* was in the press, in which were to be found essentially the same theorems derived in a different way. But Huygens had the idea of equivalent lenses which Barrow did not. When the *Dioptrica* was first published in 1703, other writers also, notably Molyneux in England, had given rules which were practically the same as those of Huygens.

Of much interest too is Huygens's way of defining and measuring the magnifying power of an optical instrument, by which he means the ratio

of the apparent size of the object as seen through the instrument to its apparent size as presented to the unaided eye. Whether or not Huygens is entitled to priority for the announcement of the fact that the magnifying power of a telescope is equal to the ratio of the focal lengths of object-glass and ocular, he attached so much importance to it that he proves it in at least three different ways in the section on telescopes and microscopes which constitutes the third part of the *Dioptrica*. Neither Kepler nor Des Cartes had understood this relation clearly. In this connection speaking of Des Cartes, Huygens says that although it is hard to believe it of a man who was so intelligent and so well informed in these matters, nevertheless he got off the track in his demonstrations of the nature and effect of the telescope and was guilty of writing things on this subject to which no meaning can be attached!

A theorem which deserves to rank as one of the most beautiful generalizations of theoretical physics, and which can be extended to the theory of radiation in general, is contained in Proposition VI in the second book of the first part of Huygens's *Tractatus de refractione et telescopiis* (see pp. 198, ff.); it may be stated as follows:

"If an object is viewed through a system of any number of lenses, and if the positions of the eye and the object are mutually interchanged without disturbing the lens-system itself, the apparent size of the object will be the same as before, and the image will be erect or inverted as before."

This will be immediately recognized as equivalent to the theorem given by Robert Smith in his *Compleat System of Opticks* (Cambridge, 1738) as the first corollary to be deduced from Roger Cotes's celebrated proposition about the "apparent distance" of an object as viewed through a system of thin lenses. Smith does not mention Huygens's name in connection with this corollary, perhaps inadvertently; for undoubtedly at that time (1738) Smith must have known of Huygens's proposition. There can be no question that Huygens is entitled to the priority here. As early as 1653 Huygens, writing to Kinner von Löwenthurm, communicated this theorem as one of the principal discoveries which he proposed to publish in his treatise on refraction and telescopes; and in 1669 he included it in the anagrams which he sent to the Royal Society. The theorem itself was not actually published until 1703, but its importance was not appreciated and it was soon forgotten or ignored. One of the many services performed by the learned editor of this volume of Huygens's works is to put this matter in its right light, because the theorem in question is one, and by no means the least, of Huygens's titles to fame. Huygens himself constantly makes use of this general principle in the solution of special problems, and modern workers will find it serviceable in the same way.

Doubtless few persons are aware nowadays that Huygens made valuable contributions to science also in the realm of physiological optics, as may be seen by looking into this volume. Most physicists are content, so to speak, to deliver radiant energy to the eye and leave it to its fate; unfortunately, comparatively few of them like Young and Helmholtz have thought it worth while to pursue the investigation further and to study the intricate phenomena of vision. Not so Huygens; he at least was keenly alive to the fact that at the other end of his microscope or telescope a

human eye was adjusted, and that visual perception is the chief thing after all. He was perfectly aware that the magnifying power of the instrument depended on the idiosyncrasies of the eye of the individual. Apparently far more accurately than Des Cartes or any of his own contemporaries, Huygens was acquainted with both the anatomical and optical structure of the eye; and he had the clearest notions about the office of the pupil and the mechanism of accommodation. The essential theory of binocular vision and depth perception was grasped by Huygens. He explains in the clearest manner how in order to see an object single with both eyes the two images on the retina must be formed at "corresponding points", although apparently he did not perceive that a solid object looks different to each eye; otherwise he might have anticipated Wheatstone and Brewster in the invention of the stereoscope. In the article on the eye and vision in the first part of the *Dioptrica* there is a description of a "simplified eye" formed by two concentric hemispheres of unequal radii. The curved surface of one of these hemispheres corresponds to the cornea and that of the other to the retina of the eye. There is a singular resemblance between Huygens's "simplified eye" and the "reduced eye" conceived by Listing in 1845, as is pointed out by the editor on page cxliv of the *Avertissement*.

After all, perhaps what impresses the reader most amid all this wealth of material is not so much the theories which are propounded and developed with such rare insight and skill, as the marvellous versatility and resourcefulness of the author and the variety of observations and experiments which underlie the whole and form the solid structure of the edifice itself. To his extraordinary mechanical ability and ingenuity Huygens owed much of his remarkable accomplishment; with him to conceive was to execute, no matter what practical obstacles might lie in the way. To this day we read with astonishment of those prodigious "aerial telescopes" with their poles and pulleys which he constructed and mounted with his own hands and with which he made some of his great discoveries in astronomy. Huygens's name is usually associated in our minds with refracting telescopes; but he devoted much study also to reflectors and preferred Newton's type of instrument to those of Gregory and Cassegrain. Many pages in the volume before us are devoted to the theory of the compound microscope and Huygens's "observations microscopiques."

But enough has been said to give the reader at least some idea of the character and scope of Huygens's optical researches. No wonder that he published comparatively little during his busy lifetime! Before he could get his thoughts safely on paper, a whole vista of new ideas begins to distract and fascinate him. New discoveries give ever a new turn to his earlier imaginings, and so he hastens onwards still eager in the pursuit of knowledge when death overtakes him at last at the summit of his great career.

Felix qui potuit rerum cognoscere causas.

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BOOKS ON RELATIVITY

Das Relativitätsprinzip. Lorentz. Einstein. Minkowski. Fortschritte der mathematischen Wissenschaften in Monographien. Herausgegeben von O. Blumenthal, No. 2. Leipzig und Berlin, B. G. Teubner, dritte Auflage, 1920. i + 146 pp.

Raum. Zeit. Materie. Von Hermann Weyl. Berlin, Julius Springer, vierte Auflage, 1921. Mit 15 Textfiguren. ix + 300 pp.

Relativity. The special and the general Theory. By Albert Einstein. Translated by Robert W. Lawson. New York, Henry Holt and Co., 1921. Frontispiece. xiii + 168 pp.

The Theory of Relativity. By Robert D. Carmichael. Mathematical Monographs, Edited by Mansfield Merriman and Robert S. Woodward, No. 12. New York, John Wiley and Sons, 2nd edition, 1920. 112 pp.

Das Relativitätsprinzip. Leichtfasslich entwickelt von Adam Angerbach. Leipzig und Berlin, B. G. Teubner, 1920. Mit 9 Figuren im Text. 57 pp.

The Concept of Nature. Tarner Lectures delivered in Trinity College, November, 1919. By A. N. Whitehead. Cambridge, The University Press, 1920. viii + 202 pp.

Wiskunde, Waarheid, Werkelijkheid. Door L. E. J. Brouwer. Groningen, P. Noordhoff, 1919. 12 pp. + 23 pp. + 29 pp.

For scientists generally, and especially for mathematicians and physicists, who understand best many of the questions involved, the theory of relativity has fundamental interest. In the following pages our purpose is to pass in review the above recent books dealing with the theory and at the same time to indicate its present state and some unsolved problems.

The collection of monographs gathered by Blumenthal begins with two papers by the Dutch physicist, Lorentz, the second and more important one of which appeared in 1904. By endeavoring to unite the classical Newtonian mechanics and the electromagnetic theory of Faraday and Maxwell into a single consistent theory, one is necessarily led to absolute space (the ether) and absolute time. In fact, physics has stood committed to absolute time since the acceptance of Newton's law of gravitation. But the experiments of Michelson in 1881 yielded an opposing result. Lorentz, in common with other physicists, had the conviction that the universe was electromagnetic in character, and he turned to the electromagnetic equations for an explanation of the difficulty. His answer to the apparent contradiction of theory and experiment was based upon the fact that the equations admitted of a transformation in which space and time were intermingled. On this basis, without giving up the concepts of absolute space and time, he was able to explain the paradox by assuming that bodies undergo a slight contraction in the direction of their motion, which for the

earth is not more than a few inches. To an observer moving at uniform velocity, the same electromagnetic equations appear to hold because such an observer uses "local time." Lorentz's explanation violated a fundamental principle, namely the pragmatic principle that no physical entity exists if its presence can never be determined by any conceivable experiment. Absolute space and time are this type of entity in his theory.

A year later in 1905, but independently, Einstein wrote his paper *Zur Elektrodynamik bewegter Körper*, which is the third paper of the collection. In this he lays the foundation of the so-called special theory of relativity. Einstein starts with a peculiarly simple type of physical universe, perhaps the simplest in harmony with all known physical laws. This is the universe of empty isotropic space in which there are infinitesimal inertia particles. The particles *appear* from such a particle to move with uniform velocity in a straight line, if observations of light signals are made with the aid of a clock. Thus the fundamental measuring instrument is the clock. It is further assumed that a light pulse appears to advance at a constant velocity from any such particle (the Michelson experiment). On the basis of these postulates the transformation equations between the coordinates set up from reference particles are deduced, and it is shown that the Maxwell electromagnetic equations are unaltered under precisely this group (the Lorentz group) of transformations. The behavior of the electron as experimentally determined is in conformity with this theory. In the short paper that follows Einstein notes that the same discussion indicates that the apparent mass of a system will depend upon its energy.

The fifth article of the collection is the mathematician Minkowski's remarkable *Raum und Zeit* of 1908. In this article, which threw a flood of light upon the work of Einstein and Lorentz, the geometry of four dimensions furnished the principal weapon. If Minkowski had lived, doubtless other equally important contributions to the theory of relativity would have come from his pen, and in any case it is clear that his influence upon Einstein can scarcely be overestimated.

The gist of Minkowski's paper is as follows: In the four-dimensional relativistic manifold of space and time, a pair of world-points or events are associated with a unique number, namely, if a particle move from the earlier world-point to the later world-point, the interval of local time elapsed will give this number. The mathematician will realize at once that we have here the elements of a non-euclidean geometry of four dimensions of simple type. The Lorentz transformations are merely the transformations of the geometry which leave this interval between world-points unaltered, and the whole theory may be subsumed in the single equation

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

where ds is the local time element, c is the velocity of light, and dx, dy, dz, dt have their customary meanings. In the same article it is pointed out how the laws of motion of the electron may be interpreted in this space; and a suitable modification of the Newtonian law of attraction is made, such as Poincaré had given earlier. There follow some instructive notes by the mathematical physicist Sommerfeld.

The remaining articles of the collection are reprints of more recent articles by Einstein, and four of them do not appear in the earlier editions. It is *Die Grundlagen der allgemeinen Relativitätstheorie* of 1916 which has aroused such widespread attention. Concerning it, Sommerfeld says in the notes just mentioned: "This general relativity theory is logically so unified and satisfactory that it has found unconditional acceptance, especially in mathematical quarters." In a paragraph added after learning of the verification of Einstein's quantitative prediction of the deviation of light by the sun, Sommerfeld says further, "The general relativity theory can therefore be regarded as an *established proposition*." If this is the truth, physical science is entering upon an era in which the new view will differ radically from the classical one.

To the mathematician, Einstein's generalized theory is of interest in several respects. In the first place it illustrates afresh the importance of taking the simplest possible case as an abstract basis of departure. Secondly, Einstein uses mathematical analogy in passing, step by step, from the simple universe of the special theory to the most general universe, and at each step the mere sense of mathematical form is sufficient to point toward a natural generalization. The mathematician may feel satisfied that the formal analogies supplied by classical dynamics and four-dimensional geometry furnish the very basis by which Einstein's generalization proceeds. And, thirdly, the technical tool which made elaboration and verification of the theory possible is the invention of the mathematicians Riemann, Christoffel, and more especially of Ricci and Levi-Civita—namely the absolute differential calculus.

What then are these successive steps of Einstein? To the writer they appear as follows:

(1) In the special theory of relativity, the universe consists of an empty isotropic space with infinitesimal inertia particles, and the central formula is that for ds^2 given above.

(2) A somewhat more general type of universe is that of a non-isotropic empty space formed by a gravitational field. It is natural, by analogy with the Riemann geometry, to assume that local time is given by a quadratic differential form ds^2 in the space and time variables, and that the particular coordinates chosen are irrelevant (general theory of relativity). But in such case only six of the ten coefficients in ds^2 must be regarded as arbitrary. Therefore there are required six equations to fix these coefficients and these conditions must be independent of the coordinate system. This leads to Einstein's conclusion that the contracted Riemann tensor vanishes. By analogy with the special theory of relativity, the paths of the particles appear as the geodesics and the paths of the light pulse satisfy the equation $ds = 0$.

(3) Still more generality is obtained if matter and energy are present. For case (1), this leads to the vanishing of the divergence of an "energy tensor." In the equations obtained in case (2) the left-hand members are tensors while the right-hand members vanish. It is natural then to assume by analogy with classical dynamics that, in case the space contains matter and energy, the right-hand member becomes the energy tensor. If we

assume this to be the case, the complete equations, as general in their scope as those of the classical theory, are obtained.

The causal principle is effective, but in an obscure form, as follows: At any "instant" for the coordinate system under consideration the time rate of change of the derivatives of the gravitational tensor formed by the system of coefficients in ds^2 and of the energy tensor are known. Also the apparent accelerations of the particles are thereby determined. Consequently it is possible to obtain the new value of the tensors and their derivatives, and the new positions and velocities of the particles, an instant later, and so to proceed indefinitely.

According to Einstein the acceleration of a particle in empty space is due to gravitational forces, and depends only on ds^2 and the coordinates chosen. The law of motion at low velocities is found to be the same as that of Newton except for very small modifications. To make a specific application, Einstein determines the necessary form of ds^2 for a single central body such as the sun, also on a postulational basis, and arrives at his brilliant predictions.

In the final paper of the collection, written in 1919, Einstein shows how the idea of a spherical space may be conveniently introduced to eliminate the difficulties due to boundary conditions.

The casual reader of the theory of relativity will feel a certain lack of concreteness. The classical physical theories seemed to touch reality in at least three ways, namely in the independent concepts of space, of time, of force. I take it to be self-evident that any genuine physical theory must touch reality somewhere. So far as I can see, the Einstein theory does this at one and at only one place, namely in its concept of local time which can be measured by means of the natural clock, the atom. In an article appearing in January, 1921, in the PROCEEDINGS OF THE BERLIN ACADEMY, Einstein lays emphasis upon this notion of the natural clock as the fundamental element, but one could wish that this had been done more definitely in his earlier articles.

From the scientific point of view the most important of the other books which we desire to review is Weyl's *Raum. Zeit. Materie.* The Einstein theory has a certain pliability in the presence of an energy tensor, which may be modified to suit the exigencies of the physical situation under discussion. On the other hand, this pliability will appear as a defect to some minds since it provides physical science with a blank form rather than with a definitive theory. It may naturally be expected that theories will be forthcoming which attempt to explain non-gravitational phenomena also on a similar quasi-geometrical basis. One recalls here the vortex theory of the atom as an analogous attempt in classical physics.

The original part of Weyl's valuable and complete treatise consists in an attempt to deduce the electromagnetic equations in such a manner. For this purpose he invents a generalization of the Riemann geometry. In the Riemann geometry, the elements ds^2 can be compared at various parts of the manifold. Weyl notes that this is a species of action at a distance and proposes to compare the elements ds^2 only for various directions at a world-point. In other words his quadratic form ds^2 is one in which

merely the ratios of the coefficients are important and the coefficients appear as undetermined up to an arbitrary multiplicative scale-factor.*

Thus in the normalization of his quadratic form by change of variables, he has five arbitrary functions (the four arbitrary coordinate functions and his scale-function) instead of the four functions available in the Einstein theory. Weyl is able to use the notion of parallel displacement due to Levi-Civita; namely, the small vector can be displaced so as to maintain size and direction in a specially chosen *geodesic* coordinate system. As this vector varies in position and returns to its starting point it will not have the same length except in the case of the Riemann geometry. The logarithmic derivative of the scale-function is a differential, whose four components behave as the components of an electromagnetic potential.

An obvious objection to Weyl's extension is that he loses contact with the real, for ds can no longer stand for the element of local time; otherwise we should expect atoms of the same element with different past histories to have different rates of vibration, and such has never been observed to be the case. In the second place, a modified ds^2 for the same manifold can be obtained which is invariant as in the Riemann geometry, and thus we are led back to the Einstein theory together with a single independent equation of the type coming under Einstein's theory. These criticisms of Weyl's work are given in the March, 1921, number of the PROCEEDINGS OF THE BERLIN ACADEMY by Einstein. Eddington has proposed a further modification of Weyl's theory in the April number of the PROCEEDINGS OF THE ROYAL SOCIETY. Eisenhart and Veblen have gone much further in an important paper in the PROCEEDINGS OF THE NATIONAL ACADEMY, February, 1922.

We pass now to the more popular treatments given in the next three books. The first of these is by Einstein himself, and affords an interesting and skillful approach to the fundamentals of the theory.

The second edition of Carmichael's book contains his earlier treatment of the special theory based upon a set of physical postulates. The new chapters are a direct summary of the results of the general theory, as presented by Einstein and Eddington. This summary is too abbreviated to be followed with much profit by the reader who has not delved elsewhere into the theory.

The tiny pamphlet by Angersbach presents a brief historical development of the notions underlying the special relativity together with an elementary presentation. It is readable.

It is obvious that the relativity theory has decided significance for philosophical thought, as indeed every new physical theory must have. In his interesting book, *The Concept of Nature*, the English philosopher-mathematician Whitehead expounds his views of the physical universe in the light of the theory of relativity. The book has obvious relations with an earlier book. † The main idea of Whitehead is that the underlying

* See his recent papers in the MATHEMATISCHE ZEITSCHRIFT, vol. 12, Nos. 1, 2 (1922).

† *An Inquiry Concerning the Principles of Natural Knowledge*, A. N. Whitehead. Cambridge, University Press, 1919. xii + 200 pp.

realities are attained by a method of extensive abstraction, a spatial point for instance being generated by all the objects of a certain category (those which include it spatially). His analysis of experience is very interesting. The mathematician will regret frequently redundancy and vagueness in philosophical treatises. Of this there is little in Whitehead's book. The importance and the exactitude of many of his analyses must be admitted. Maxime Bôcher once said to the writer, "What man would be a philosopher who might be a mathematician!" One feels that Mr. Whitehead deserves both titles.

The contrasting account given in this book between the old theories and the new theory of relativity is interesting. Characterizing the old theories, Whitehead says: "For example, colour is the result of a transmission from the material object to the perceiver's eye; and what is thus transmitted is not colour. Thus colour is not part of the reality of the material object. Similarly for the same reason sounds evaporate from nature. Also warmth is due to the transfer of something which is not temperature. Thus we are left with spatio-temporal positions, and what I may term the 'pushiness' of the body. This leads us to eighteenth and nineteenth century materialism, namely, the belief that what is real in nature is matter, in time and in space and with inertia."

The new relativity theory he expounds as follows: "Let us make therefore the general statement that four measurements, respectively of independent types (such as measurements of lengths in three directions and a time) can be found such that a definite event-particle is determined by them in its relations to other parts of the manifold. . . . If (p_1, p_2, p_3, p_4) be a set of measurements of this system, then the event-particle which is thus determined will be said to have p_1, p_2, p_3, p_4 as its coordinates in this system of measurement." . . . "Then we should naturally say that (p_1, p_2, p_3) determined a point in space and that the event particle happened at that point at the time p_4. . . . Furthermore the inhabitant of Mars determines event-particles by another system of measurements. Call his system the q-system. According to him, (q_1, q_2, q_3, q_4) determines an event-particle, and (q_1, q_2, q_3) determines a point and q_4 a time. But the collection of event-particles which he thinks of as a point is entirely different from any such collection which the man on earth thinks of as a point. Thus the q-space for the man on Mars is quite different from the p-space for the land-surveyor on earth. . . ."

". . . We have got to find the way of expressing the field of activity of events in the neighborhood of some definite event-particle E of the four-dimensional manifold. I bring in a fundamental physical idea which I call the 'impetus' to express this physical field. The event-particle E is related to any other neighboring event-particle P, by an element of impetus." ··· "Einstein showed how to express the characters of the assemblage of elements of impetus of the field surrounding an event-particle E in terms of ten quantities which I will call $J_{11}, J_{12}, \ldots$ The numerical values of the J's will depend on the system of measurement adopted, but are so adjusted to each particular system that the same value is obtained for the element of impetus between E and P, whatever be the system of measurement

adopted. This fact is expressed by saying that the ten J's form a 'tensor.' " . . . "We now return to the path of the attracted particle. We add up all the elements of impetus in the whole path, and obtain thereby what I call the 'integral impetus.' The characteristic of the actual path as compared with neighbouring alternative paths is that in the actual paths the integral impetus would neither gain nor lose, if the particle wobbled out of it into a small extremely near alternative path."

Evidently Whitehead is expressing the relativistic theory of the path of a particle in a gravitational field. The indefinite general terms used stand in unfavorable contrast with those which can be used in the exposition of the classical theory.

Finally we turn to the little pamphlet by Brouwer. Many American mathematicians have read in this BULLETIN (November, 1913) a translation of Brouwer's paper on *Intuitionism and Formalism*, which is the final essay of the pamphlet. Those who know the mathematical work of Brouwer will be interested in these essays, in the second of which he touches upon the special theory of relativity with emphasis upon the notion of *group*. Attention should also be directed to his noteworthy analysis of the logical principle of the *excluded middle*, to which reference is made in the first essay.

In conclusion, one or two remarks of general character suggest themselves.

The theory of relativity in its general form or in its special form involves a definite group. To the mathematician at least it would be of considerable interest to see physical theories developed for other special groups. In particular the most general group of all, that of *analysis situs*, suggests itself, for this alone appears strictly proper in the general theory of relativity, where any transformation of coordinates whatsoever ought to be admitted. If such a theory can be constructed, the interrelation of continuous manifolds of arbitrary form will form the essential element. A theory of this kind would seem to be consonant with quantum theory.

Also, with others, we may call attention to the fact that no theory of relativity so far explains the difference between positive and negative electricity, or throws any light upon the constitution of matter. Furthermore no real reason appears why the velocities of the stars relative to one another are so small in comparison with the velocity of light.

While awaiting further developments, let us at least say with Whitehead of Einstein's investigations: "They have made us think." Some may agree with the final words of Weyl: "A few chords of that harmony of the spheres of which Pythagoras and Kepler dreamed have fallen upon our ears."

G. D. BIRKHOFF.

SHORTER NOTICES

Einführung in die theoretische Physik. Von Clemens Schaefer. Band II, Teil 1. *Theorie der Wärme, Molekular-kinetische Theorie der Materie.* Berlin, Vereinigung wissenschaftlicher Verleger, 1921. x + 562 pp.

The first volume on mechanics (particles, rigid bodies and continua) when reviewed in these pages was signalized for its excellence of material and of style. The second volume continues the excellences of the first. The rest of the work will be awaited with interest, and, by those engaged in teaching theoretical physics, with impatience. It is a pity we cannot have such books in English. Twenty years ago there was hope that Andrew Gray's *Treatise on Physics* (of which vol. I, *Dynamics*, alone appeared) would fill this lacuna. Schaefer has written his present volume since the Armistice. He has been at Breslau and is now at Marburg.

There are eleven chapters as follows: 1. Heat conduction. 2. First Law of thermodynamics. 3. Second Law of thermodynamics. 4. Homogeneous systems. 5. Heterogeneous systems. 6. Special systems (gases and dilute solutions). 7. Chemical affinity and Nernst's theorem. 8. Kinetic theory of gases. 9. Entropy and probability. 10. Statistical mechanics. 11. Quantum theory.

The arrangement is logical. General old-fashioned theories of heat conduction and thermodynamics come first—general thermodynamics, be it noted, not the special idealized systems. The treatment of chemical affinity is more modern but at least in its main lines is now firmly established. The last four statistical chapters are excellent: no over-elaboration on ancient kinetic theory, a clear treatment of the fundamentals of statistical mechanics, and an excellent and convincing presentation of quantum theory so far as heat phenomena are concerned. Nothing is said of radiation—not even the Stefan law which sometimes is deduced from a Carnot cycle and could have been inserted in Chapter III as illustrative material.

The last few pages on Nernst's theory of the degeneration of ideal gases (Entartung) should have been omitted in a book of this kind. It is too special and not yet well enough substantiated either experimentally or theoretically (as the author notes). The space thus gained could well have been given to real gases, under which only van der Waals's equation is given, whereas for some not remote purposes other equations are useful; and for accurate descriptions of the behavior of substances over long ranges of temperature, pressure, and density, Keyes's equation is incomparably superior to van der Waals's.

Generally the mathematical treatment is elegant—not the elegance of the mathematical virtuoso, but that of the working physicist. (A notable exception is the painful summation, p. 502, of the series $\Sigma n x^n$ in place of its evaluation as the derivation of Σx^n multiplied by x.) In the main, physical concepts are kept well in the foreground throughout mathematical

discussions, and physical philosophy is delightfully in evidence throughout involved physical discussions. Tables of numerical data help at times to increase the effective reality of the work. One thing, however, continental authors disregard to their serious detriment, and that is exercises for the student. When exercises are well selected they double the educational value of any book. Schaefer has the teacher's instincts and could easily have furnished excellent exercises.

It would be only too easy to stretch this review to great lengths, mentioning such clear-cut analysis as that of Boltzmann's "ergodische" systems with its unusual frankness in pointing out that there are no such systems (p. 439), of the total separation of Liouville's theorem and the equipartition of energy, etc., of the discussion of anomalies (Schwankungen) and its bearing on the work of Perrin and his followers. But there are too many details to cite, and we had best not begin. Let the book be widely read.

EDWIN B. WILSON.

The Elements of Non-Euclidean Geometry. By D. M. Y. Sommerville. Chicago, The Open Court Publishing Company; and London, G. Bell and Sons, 1919.* xvi + 274 pp.

The printing of a second edition of this book, as well as the fact that it now, for the first time, appears as a publication of the Open Court Publishing Company, speaks well for the attention it has received from mathematical readers.

A careful comparison of the two editions shows no changes. The very few typographical errors and the somewhat more frequent unintended slips in the text remain. For instance, on p. 10, l. 20, a parenthesis is still lacking; likewise an "s" on p. 22, l. 1. There remains (p. 54) this remarkable series of statements:—"A triangle has therefore four circumcircles. ··· There cannot be more than one real circumcentre" [and may, of course, be none]. "This point, which we may call *the* circumcentre, ··· may be real, at infinity, or ideal." On p. 204, line 19, R is still called the earth's radius, although the radius of the earth's orbit is clearly meant.

It is to be sincerely hoped that the popularity of the book may before long require still another printing, and that a careful revision may then make it as accurate as it is interesting.

EDWARD S. ALLEN.

Annuaire du Bureau des Longitudes pour 1921. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1921. 7 + 710 + 130 pp.

Before 1904, each issue of the Annuaire contained all the information which the Bureau considered necessary for publication. Owing to the number of pages, which had gradually increased, it was then divided and certain articles in which there was little or no annual change were given only in alternate years. The volume has now grown again to a size which is near its maximum for convenient handling and one finds references which include the previous four volumes. Its chief annual feature is the full

* Originally published in 1914 by G. Bell and Sons, and reviewed in this BULLETIN, vol. 21, May, 1915, by J. L. Coolidge.

astronomical information for the current year and a briefer summary for the succeeding year. There are usually one or more appendices containing articles on various topics. This year M. Bigourdan writes on proper motions and radial velocities of stars—a summary which, starting at the beginning, takes the reader to the work which is now in progress at many observatories. A valuable feature is the full index of all matters contained in the current issue with references, where necessary, to the previous four volumes.

ERNEST W. BROWN.

Die Grundlagen der Geometrie als Unterbau für die Analytische Geometrie. By Lothar Heffter. Leipzig and Berlin, B. G. Teubner, 1921. ii + 27 + viii pp.

The author's purpose is to provide a foundation for the study of analytic geometry and this he proposes to do by stating a set of projective postulates sufficient for the introduction of the double ratio, the fundamental concept of the projective scale. He explicitly states his intention to ignore the question of the independence of his postulates—and would have us regard them as "one possible way of idealizing the results of observation" rather than as a purely abstract gift from heaven. His undefined elements are point, line, plane and incidence. He lists twenty "axioms of alignment" (Verknüpfung), arranged in ten pairs of dual propositions, three "axioms of order" (separation), and one of continuity. To secure the restriction to the affine and then to the euclidean metric geometries, he introduces one axiom of parallelism and one axiom of orthogonality. To those interested in a brief formulation of the postulational foundation, from a projective point of view, for the study of analytic geometry the little pamphlet will be of value.

J. W. YOUNG.

Les Théories d'Einstein. By Lucien Fabre. Paris, Payot et Cie., 1921. 225 pp.

This book will be found of interest because of its valuable survey of the historical background of Einstein's theories. The author sometimes fails to define the symbols used. (For example, on pages 161, 179, and 243, and on page 196 there are incorrect definitions.) In spite of these defects, however, the book is one of the best elementary expositions of Einstein's work that has been published.

C. N. REYNOLDS.

NOTES

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 23, the following officers were elected for the year beginning June, 1922: Chairman, William Duane; Secretary, W. E. Tisdale; Executive Committee, G. D. Birkhoff, William Bowie, C. E. Mendenhall, A. Trowbridge. The following persons were elected members-at-large: G. A. Bliss, F. A. Saunders, A. Trowbridge. New members representing scientific societies were announced as follows: for the American Astronomical Society, E. B. Frost; for the American Mathematical Society, C. N. Haskins; for the American Physical Society, H. G. Gale and C. E. Mendenhall.

The annual meeting of the National Council of Mathematics Teachers was held at Chicago on March 1, 1922, in connection with the meeting of the Department of Superintendence of the National Education Association. A number of papers were read, dealing with the teaching of secondary mathematics.

Professor W. F. Osgood and G. D. Birkhoff have been elected corresponding members of the Göttingen Academy of Sciences.

Professor M. Planck has been elected a foreign member of the Swedish Academy of Sciences.

Professor A. Gutzmer has been elected president of the Leopoldinisch-Carolinische Akademie der Naturforscher, as successor to Professor A. Wangerin, who has resigned. Professors H. Hahn and R. Schüssler have been elected members of this Academy.

Professor E. I. Fredholm, of the University of Stockholm, has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor H. A. Schwarz.

Requests to authors of mathematical works for reprints have been sent by L'Istituto Matematico della R. Università di Bologna.

Cambridge University has awarded a Rayleigh prize to T. A. Brown, of Trinity College, for an essay *On a class of factorial series.*

The honorary degree of doctor of science has been conferred on Sir Thomas Muir by the University of Cape Town, in recognition of his researches in mathematics and the history of mathematics.

Professor A. S. Eddington has been elected president of the Royal Astronomical Society.

In connection with the announcement of the distribution of reprints of articles by the late Professor Maxime Bôcher (this BULLETIN, vol. 28

(1922), p. 74), it is stated that the reprints available, numbered as in this BULLETIN, vol. 25 (1919), p. 209, are as follows: 1, 3, 4, 6, 7, 9, 12, 16, 17, 18, 21, 23, 25, 26, 30–34, 36–42, 48, 51, 52, 53, 56, 59, 60, 63, 64, 66–69, 74, 78, 79, 83, 84, 85, 87–92, 97, 99, 100. Reprints will be sent, while the supply lasts, to those applying by number for the ones desired to Professor O. D. Kellogg, 20 Craigie St., Cambridge 38, Mass. The recipient will be expected to defray mailing costs; no other charge will be made except for No. 67, *Introduction to the Theory of Fourier Series* ($0.75) and No. 99, *Leçons sur les Méthodes de Sturm* ($1.25).

Professor G. D. Birkhoff will deliver four lectures on the theory of relativity, and four lectures on the problem of three bodies, at Stanford University during the latter part of June, 1922.

Dr. H. Brandt, of the Karlsruhe Technical School, has been appointed to a professorship at the Technical School at Aachen.

Dr. W. Schmeidler, of the University of Kiel, has been appointed to a professorship at the Breslau Technical School.

Dr. O. Szàsz has been promoted to an associate professorship at the University of Frankfurt a. M.

Dr. F. Bernstein has been promoted to a full professorship at the University of Göttingen.

Professor J. Radon, of the University of Hamburg, has been appointed full professor of mathematics at the University of Greifswald.

Dr. G. Prange, of the University of Halle, has been appointed to a full professorship at the Hannover Technical School.

Professor F. Noether has been appointed associate professor of applied mathematics at the University of Heidelberg.

Dr. Pohlhausen has been promoted to an associate professorship of applied mathematics at the University of Rostock.

At the Stuttgart Technical School, Dr. K. Kommerell has been promoted to an associate professorship, and Dr. F. Pfeiffer, of the University of Heidelberg, has been appointed full professor.

Dr. R. Kõnig has been promoted to a full professorship of mathematics at the University of Tübingen.

In the Faculty of Sciences of the University of Paris, Professor P. Montel has been appointed to the professorship of general mathematics, and Professor Leduc to the professorship of theoretical physics, as successors, respectively, to Professors J. Drach and E. Cotton, called to other professorships in the same faculty. The title of the chair of "differential and integral calculus" has been changed to "the theory of groups and the calculus of variations."

Mr. A. L. Dixon, fellow and tutor of Merton College, has been appointed Wayneflete professor of pure mathematics at Oxford University, as successor to Professor E. B. Elliott.

Dr. G. B. Jeffery has been appointed to the University chair of mathematics at King's College, London University.

Professor W. F. Osgood has been appointed acting dean of the graduate school of arts and sciences at Harvard University for the second half of the present academic year.

At the University of Rochester, Professor A. S. Gale has been appointed Freshman Dean.

At Yale University, Assistant Professor Leigh Page, of the department of physics, has been promoted to a full professorship of the mathematical sciences, with assignment to the Sheffield Scientific School. Dr. W. L. Crum has been promoted to an assistant professorship of mathematics. Professor L. S. Hill, of the University of Maine, has been appointed instructor in mathematics.

Dr. E. B. Wilson, professor of mathematical physics at the Massachusetts Institute of Technology, has been appointed professor of vital statistics at Harvard University.

Assistant Professor C. G. P. Kuschke has been promoted to be full professor and head of the department of mathematics at the University of Porto Rico.

At the University of Kansas, for the year 1920–21, Dr. U. G. Mitchell was advanced to a full professorship and Dr. Guy Smith was appointed assistant professor; for the year 1921–22, Dr. E. B. Stouffer was advanced to a full professorship.

Dr. H. C. M. Morse and Dr. W. L. G. Williams have been promoted to assistant professorships at Cornell University.

Professor G. Kohn, of the University of Vienna, died December 15, 1921, at the age of sixty-two years.

Professor E. Jahnke, of the Berlin Technical School, died October 18, 1921, at the age of fifty-eight years.

Professor L. Koenigsberger, of the University of Heidelberg, died December 15, 1921, at the age of eighty-three years.

Professor M. Noether, of the University of Erlangen, died December 13, 1921, at the age of seventy-seven years.

Dr. G. B. Mathews, formerly professor of mathematics at the University College of North Wales, died March 19, 1922, at the age of sixty-one years.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ARCHIMEDES. See VER EECKE (P.).

ARWIN (A.). Die Kongruenzen $(\lambda + 1)^p \equiv 0 \pmod{p^p}$, und die Natur ihrer Lösungen. Leipzig, 1921. 4to. 38 pp.

BAKER (H. F.). Principles of geometry. Volume 1: Foundations. Cambridge, University Press, 1922. 8vo. 12 + 184 pp.

BLASCHKE (W.). Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. I: Elementare Differentialgeometrie. (Die Grundlehren der mathematischen Wissenschaften, herausgegeben von R. Courant, Band 1.) Berlin, Springer, 1921. 10 + 230 pp.

BOREL (E.). Leçons sur les fonctions entières. 2e édition. Paris, Gauthier-Villars, 1921. 9 + 161 pp.

CONGRÈS INTERNATIONAL. See VILLAT (H.).

COURANT (R.). See BLASCHKE (W.), KNOPP (K.).

DE MORGAN (A.). A budget of paradoxes. Edited by D. E. Smith. 2d edition. Chicago and London, Open Court, 1921. 2 volumes.

DEUTSCHBEIN (M.). Wahrscheinlichkeit und Induktion. (Diss., Halle.) Cöthen, Verlag von Otto Schulze, 1920.

EHRENFELS (C.). Das Primzahlengesetz entwickelt und dargestellt auf Grund der Gestalttheorie. Leipzig, Reisland, 1922.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 3, Heft 5–6: K. Hensel, Arithmetische Theorie der algebraischen Funktionen. H. W. E. Jung, Arithmetische Theorie der algebraischen Funktionen zweier unabhängigen Veränderlichen. Leipzig, Teubner, 1921.

EULER (L.). Opera omnia. Series I: Opera mathematica. Volumen VI: Commentationes algebraicae ad theoriam aequationum pertinentes. Ediderunt F. Rudio, A. Krazer, P. Stäckel. Leipzig, Teubner, 1921. 20 + 509 pp.

FISHER (A.). The mathematical theory of probabilities. Volume 1. Reissue, with new chapters. New York, Macmillan, 1922.

FRICKE (R.). Die elliptischen Funktionen und ihre Anwendungen. Teil 2: Die algebraischen Ausführungen. Leipzig, Teubner, 1922. 8vo. 8 + 546 pp.

GRÜNBERG (J.). Die "D"-Linien der Mittelpunktsflächen zweiten Grades. (Diss., Heidelberg.) Göttingen, Dietrichsche Universitäts-Buchdruckerei, 1920.

HEATH (T.). A history of Greek mathematics. Volume 1: From Thales to Euclid. Volume 2: From Aristarchus to Diophantus. Oxford, Clarendon Press, 1921. 15 + 446 + 11 + 586 pp.

VON HELMHOLTZ (H.). Schriften zur Erkenntnistheorie. Herausgegeben und erläutert von P. Hertz und M. Schlick. Berlin, Springer, 1921. 10 + 176 pp.

HENSEL (K.). See ENCYKLOPÄDIE.

HERTZ (P.). See VON HELMHOLTZ (H.).

HUMBERT (P.). Introduction à l'étude des fonctions elliptiques. Paris, Hermann, 1922. 8vo.

JOHNSON (W. E.). Logic. Part 1. Cambridge, University Press, 1921. 40 + 255 pp. 15s.

JUNG (H. W. E.). See ENCYKLOPÄDIE.

KEYSER (C. J.). Mathematical philosophy. A study of fate and freedom. Lectures for educated laymen. New York, Dutton, 1922. 14 + 466 pp. $4.70

KLAUCKE (F.). Ueber Raumkurven, zwischen deren beiden Krümmungen eine Beziehung besteht. (Diss., Halle.) Halle, H. John, 1916. 128 pp.

KNOPP (K.). Theorie und Anwendung der unendlichen Reihen. (Die Grundlehren der mathematischen Wissenschaften, herausgegeben von R. Courant, Band 2.) Berlin, Springer, 1922. 10 + 474 pp.

KOSCHMIEDER (L.). Untersuchungen über Jacobischen Polynome. Habilitationsschrift. Breslau, Trewendt und Granier, 1919.

KRAZER (A.). See EULER (L.).

LINDEMANN (F.). Die Biegungsflächen einer gegebenen Fläche. München, 1921.

LOBATTO (—.). See SCHUH (F.).

MALET (H.). Etude géométrique des transformations birationnelles et des courbes planes. Paris, Gauthier-Villars, 1921. 8vo. 262 pp.

VON MANGOLDT (H.). Einführung in die höhere Mathematik für Studierende und zum Selbstudium. 1ter und 2ter Band, 3te Auflage. 3ter Band, 2te Auflage. Leipzig, 1921.

MARCHAND BEY (E. E.). Mathématiques inédites. Etude complète des spirales logarithmiques par nos méthodes élémentaire et analytique. Epiry, chez l'auteur, et Paris, Desforges, 1921. 8vo. 87 pp.

NEVILLE (E. H.). The fourth dimension. Cambridge, University Press, 1921. 8vo. 56 pp.

PASCAL (E.). Repertorium der höheren Mathematik. 2ter Band, 2te Hälfte: Raumgeometrie. Zweite völlig umgearbeitete Auflage der Deutschen Ausgabe, herausgegeben von H. E. Timmerding. Leipzig, Teubner, 1922. 12 + 629 pp.

PLATEN (C.). Beiträge zur Untersuchung der harmonischen Kovarianten zweier Kurven. (Diss., Halle.) Halle, Hohmann, 1920. 60 pp.

ROUGIER (L.). Les paralogismes du rationalisme; essai sur la théorie de la connaissance. Paris, Alcan, 1921. 8vo.

RUDIO (F.). See EULER (L.).

RUSSELL (B.). The analysis of mind. New York, Macmillan, 1921. 310 pp.

SCHILLING (B.). Ueber eine Berührungstransformation, die den Punkten des einen Feldes Geradenpaare zuordnet. (Diss., Dresden.) Borna-Leipzig, Druck von R. Noske, 1919.

SCHLICK (M.). See VON HELMHOLTZ (H.).

SCHUH (F.). Lessen over de hoogere algebra. Eerste deel. Uitgegeven als negende druk van Lobatto's Lessen over hoogere algebra. Groningen, Noordhoff, 1921. 20 + 526 pp.

SMITH (D. E.). See DE MORGAN (A.).

STÄCKEL (P.). See EULER (L.).

STUDY (E.). Denken und Darstellung. Logik und Werte. Dingliches und Menschliches in Mathematik und Naturwissenschaften. (Sammlung Vieweg.) Braunschweig, Vieweg, 1921. 4 + 43 pp.

DE TANNENBERG (W.). Conférences sur les transformations de géométrie plane. Paris, Vuibert, 1921. 8vo. 51 pp.

TIMERDING (H. E.). See PASCAL (E.).

TROPFKE (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. Band 2: Allgemeine Arithmetik. 2te, verbesserte und sehr vermehrte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921.

VEBLEN (O.). The Cambridge Colloquium, 1916. Part II: Analysis situs. New York, American Mathematical Society, 1922. 8vo. 10 + 150 pp. Paper. $2.00; to members of the Society, $1.50

VER EECKE (P.). Les œuvres complètes d'Archimède, traduites du grec en français avec une introduction et des notes. Paris et Bruxelles, Desclée, de Brouwer et Cie., 1921. 8vo. 60 + 554 pp.

VILLAT (H.). Comptes rendus du Congrès international des mathématiciens (Strasbourg, 22–30 Septembre 1920). Toulouse, Privat, 1921. 670 pp.

II. ELEMENTARY MATHEMATICS

BECK (E. G.). Real mathematics. Intended mainly for practical engineers, as an aid to the study and comprehension of mathematics. (Oxford Technical Publications.) London, Henry Frowde, and Hodder and Stoughton, 1922. 10 + 306 pp.

CHADWICK (W.). Number, weight and fractional calculator. 6th edition. London, Lockwood, 1921. 870 pp.

DICKSON (L. E.). A first course in the theory of equations. New York, Wiley, 1922. 6 + 168 pp. $1.75

DOUGAL (W. J.). See LAURIE (J.).

FARNSWORTH (P. V.). Industrial mathematics practically applied. An instruction and reference book for students in manual training, industrial, and technical schools, and for home study. New York, Van Nostrand, 1921. 285 pp. $2.50

GHEURY DE BRAY (M. E. J.). Exponentials made easy; or, the story of "Epsilon." London, Macmillan, 1921. 10 + 253 pp. 4s. 6d.

GRÜNBAUM (H.). Funktionenlehre und Elemente der Differential und Integralrechnung. 5te, erweiterte Auflage, neu bearbeitet von S. Jakobi. Leipzig, 1921.

HUMBERT (E.). Problèmes d'algèbre et exposé des principales théories. Paris, Vuibert, 1920. 8vo. 452 pp.

JAKOBI (S.). See GRÜNBAUM (H.).

KOWALEWSKI (G.). Mathematica delectans. Ausgewählte Kapitel aus der Mathematik der Spiele in gemeinverständlicher Darstellung. Heft 1: Boss Puzzle und verwandte Spiele. Leipzig, Engelmann, 1921. 8vo. 72 pp.

KRATHWOHL (W. C.). See PALMER (C. I.).

LAURIE (J.). High rate interest tables. Edited by W. J. Dougal. London, Routledge, 1921. 8vo. 21s.

MACMAHON (P. A.). New mathematical pastimes. Cambridge, University Press, 1921. 10 + 116 pp.

MARSHALL (W. C.). Graphical methods for schools, colleges, statisticians, engineers and executives. New York, 1921. 259 pp. $3.00

MOREUX (T.). Petit formulaire mathématique. Tables de logarithmes et tables diverses. Paris, Doin, 1922.

PALMER (C. I.) and KRATHWOHL (W. C.). Analytic geometry, with introductory chapter on the calculus. New York, McGraw-Hill, 1921. 14 + 347 pp.

ROSENBERGER (N. B.). The place of the elementary calculus in the senior high school mathematics; and suggestions for a modern presentation of the subject. (Columbia University Contributions to Education, No. 117.) New York, Columbia University Press, 1921. 7 + 81 pp.

SCHMALL (C. N.). Analytical geometry. New York, Van Nostrand, 1921. 11 + 338 pp.

WOLFE (W. S.). Graphical analysis. New York, 1921. 374 pp. $4.00.

III. APPLIED MATHEMATICS

ANNUAIRE pour l'an 1922, publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1922. 16mo. 8 + 800 pp.

ARRHENIUS (S.). Le destin des étoiles. Etudes d'astronomie physique. Traduction française par T. Seyrig. Paris, Alcan, 1921. 16mo. 224 pp.

BANKI (D.). Energie-Umwandlungen in Flüssigkeiten. In 2 Bänden. Band 1: Einleitung in die Konstruktionslehre der Wasserkraftmaschinen, Kompressoren, Dampfturbinen und Aeroplane. Berlin, Springer, 1921.

BIGOURDAN (G.). Gnomonique ou traité théorique et pratique de la construction des cadrans solaires. Paris, Gauthier-Villars, 1922. 8vo. 214 pp.

BRAGG (W.). Electrons and ether waves. Being the twenty-third Robert Boyle lecture, Oxford University Junior Scientific Club. London, Oxford University Press, 1921. 14 pp.

CAMPBELL (N. R.). Modern electrical theory. Supplementary chapters. Chapter 15: Series spectra. Cambridge, University Press, 1921. 8 + 110 pp.

CARSLAW (H. S.). Introduction to the mathematical theory of the conduction of heat in solids. Volume 2 of the second edition of Introduction to the theory of Fourier's series and integrals and the mathematical theory of the conduction of heat. London, Macmillan, 1921. 8vo. 12 + 268 pp.

CHINI (M.). Lezioni sull'integrazione delle equazioni differenziali ad uso degli studenti di chimica. Livorno, Giusti, 1921.

DINGLER (H.). Kritische Bemerkungen zu den Grundlagen der Relativitätstheorie. Leipzig, Hirzel, 1921.

DE DONDER (T.). La gravifique einsteinienne. Paris, Gauthier-Villars, 1921.

DURAND (W. F.). Hydraulics of pipe lines. London, Constable, 1921. 16 + 271 pp. 18s.

EGERER (H.). Ingenieur-Mathematik. Lehrbuch der höheren Mathematik für die technischen Berufe. 1ter Band, berichtigter Neudruck, und 2ter Band. Berlin, Springer, 1921–22.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band V 2, Heft 4: W. Pauli, Relativitätstheorie. Leipzig, Teubner, 1921. 237 pp. (Also published separately, as Sonderabdruck.)

ENSSLIN (M.). Elastizitätslehre für Ingenieure. 1ter Teil. 2te, verbesserte Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 147 pp.

FISCHER (P. B.). Einführung in die darstellende Geometrie. (Aus Natur und Geisteswelt.) Leipzig, Teubner, 1921.

FRANCK (M.). La loi de Newton est la loi unique. Théorie mécanique de l'univers. Paris, Gauthier-Villars, 1921. 8vo. 158 pp.

FRIEDRICH (K.). Neue Grundlagen und Anwendungen der Vektorrechnung. München, Oldenbourg, 1921.

GANDILLOT (M.). Ether ou relativité. Paris, Gauthier-Villars, 1922. 16mo. 14 + 84 pp.

GRAY (A.). Absolute measurements in electricity and magnetism. 2d edition, rewritten and extended. London, Macmillan, 1921. 19 + 837 pp. 42s.

GUYE (C. E.), RATNOWSKY (S.) et LAVANCHY (C.). Vérification expérimentale de la formule de Lorentz-Einstein. Genève, Kundig, 1921. 4to. 92 pp.

HEFFTER (L.). Ueber eine vierdimensionale Welt. Gemeinverständliche Einführung in die Relativitätstheorie. 2te, gänzlich umgearbeitete Auflage. Freiburg i. B., Speyer und Kaerner, 1921.

HIGBEE (F. G.). Descriptive geometry. New York, 1921. $1.50

HOLZER (H.). Die Berechnung der Drehschwingungen und ihre Anwendung im Maschinenbau. Berlin, Springer, 1921.

JANET (P.). Problèmes et exercices d'électricité générale. Paris, Gauthier-Villars, 1921. 8vo. 255 pp.

KIRSCH (B.). Vorlesungen über technische Mechanik elastischer Körper. Wien, 1919.

KÖNIG (H.). Die Bewegung des rotierenden Langgeschosses. Göttingen, 1919.

LAMB (G. C.). Alternating currents. Parts 1–2. Cambridge, University Press, 1921. 8 + 73 + 8 + 127 pp. 5s. 6d. + 7s. 6d.

VON LAUE (M.). Die Relativitätstheorie. Band 2. Braunschweig, 1921.

LAVANCHY (C.). See GUYE (C. E.).

LECORNU (L.). See LÉMERAY (E. M.).

LÉMERAY (E. M.). L'éther actuel et ses précurseurs. Préface de L. Lecornu. Paris, Gauthier-Villars, 1922. 16mo. 141 pp.

LENARD (P.). Ueber Aether und Uräther. Leipzig, Hirzel, 1921. 56 pp.

PAULI (W.). See ENCYKLOPÄDIE.

RATNOWSKY (S.). See GUYE (C. E.).

SEYRIG (T.). See ARRHENIUS (D.).